Parton distribution functions in QED

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14th APCTP-BLTP JINR Joint Workshop

14th July 2023

Outline

Motivation

Motivation:

- Development of physical programs for future high-energy e^+e^- colliders
- Having high-precision theoretical description of basic e^+e^- processes is of crucial importance
- Two-loop calculations are in progress, but higher-order QED corrections are also important
- The formalism of QED parton distribution functions can give a fast estimate of the bulk of higher-order effects

Future e^+e^- collider projects

Linear Colliders

- ILC, CLIC
- ILC: technology is ready,

not to be built in Japan (?)

E_{tot}

- ILC: 91; 250 GeV — 1 TeV
- CLIC: 500 GeV — 3 TeV

$\mathcal{L} \approx 2 \cdot 10^{34}$ cm^{-2}s^{-1}

Stat. uncertainty $\sim 10^{-3}$

Circular Colliders

- FCC-ee, TLEP
- CEPC
- $\mu^+\mu^-$ collider (μTRISTAN)

E_{tot}

- 91; 160; 240; 350 GeV

$\mathcal{L} \approx 2 \cdot 10^{36}$ cm^{-2}s^{-1} (4 exp.)

Stat. uncertainty $\sim 10^{-6}$

Tera-Z mode!

Super Charm-Tau Factory Projects

Budker Institute of Nuclear Physics + Sarov and/or China

Colliding electron-positron beams with c.m.s. energies from 2 to 7 GeV with unprecedented high luminosity $10^{35} cm^{-2} c^{-1}$

The electron beam will be longitudinally polarized

The main goal of experiments at the Super Charm-Tau Factory is to study the processes charmed mesons and tau leptons, using a data set that is 2 orders of magnitude more than the one collected by BESIII

Estimated experimental precision

Now:

Quantity	Theory error	Exp. error
M_W [MeV]	4	15
$\sin^2\theta^l_{eff}[10^{-5}]$	4.5	16
Γ_Z [MeV]	0.5	2.3
$R_b[10^{-5}]$	15	66

Quantity	ILC	FCC-ee	CEPC	Projected theory error
M_W [MeV]	3–4	1	3	1
$\sin^2\theta^l_{eff}[10^{-5}]$	1	0.6	2.3	1.5
Γ_Z [MeV]	0.8	0.1	0.5	0.2
$R_b[10^{-5}]$	14	6	17	5–10

The estimated error for the theoretical predictions of these quantities is given, under the assumption that $O(\alpha\alpha_s^2)$, fermionic $O(\alpha^2\alpha_s)$, fermionic $O(\alpha^3)$, and leading four-loop corrections entering through the ρ-parameter will become available.

Perturbative QED (I)

Fortunately, in our case the general perturbation theory can be applied:

$$\frac{\alpha}{2\pi} \approx 1.2 \cdot 10^{-3}, \quad \left(\frac{\alpha}{2\pi}\right)^2 \approx 1.4 \cdot 10^{-6}$$

Moreover, other effects: hadronic vacuum polarization, (electro)weak contributions, hadronic pair emission, etc. are small in, e.g., Bhabha scattering and can be treated one-by-one separately

Nevertheless, there are some enhancement factors:

1) First of all, the large logarithm $L \equiv \ln \frac{\Lambda^2}{m_e^2}$ where $\Lambda^2 \sim Q^2$ is the momentum transferred squared, e.g., $L(\Lambda = 1\,\text{GeV}) \approx 16$ and $L(\Lambda = M_Z) \approx 24$.

2) The energy region at the Z boson peak ($s \sim M_Z^2$) requires a special treatment since factor M_Z/Γ_Z appears in the annihilation channel

Perturbative QED (II)

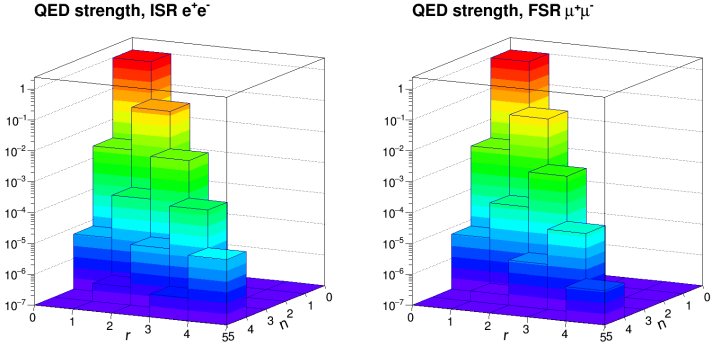


Fig.: The parameter γ_{nr} characterizing the size of the QED corrections,

$$\gamma_{nr} = \left(\frac{\alpha}{\pi}\right)^n \left(2\ln\frac{M_Z^2}{m_f^2}\right)^r, \qquad 1 \le r \le n$$

Figure from [S.Jadach and M.Skrzypek, arXiv:1903:09895]

Perturbative QED (III)

Methods of resummation of QED corrections

- Resummation of vacuum polarization corrections (geometric series)
- Yennie–Frautschi–Suura (YFS) soft photon exponentiation and its extensions, see, e.g., PHOTOS
- Resummation of leading logarithms via QED structure functions or QED PDFs (E.Kuraev and V.Fadin 1985; A. De Rujula, R. Petronzio, A. Savoy-Navarro 1979)

N.B. Resummation of real photon radiation is good for inclusive observables...

Leading and next-to-leading logs in QED

The QED leading (LO) logarithmic corrections

$$\sim \left(\frac{\alpha}{2\pi}\right)^n \ln^n \frac{s}{m_e^2}$$

were relevant for LEP measurements of Bhabha, $e^+e^- \to \mu^+\mu^-$ etc. for $n \leq 3$ since $\ln(M_Z^2/m_e^2) \approx 24$

NLO contributions

$$\sim \left(\frac{\alpha}{2\pi}\right)^n \ln^{n-1} \frac{s}{m_e^2}$$

with $n = 3$ are required for future e^+e^- colliders

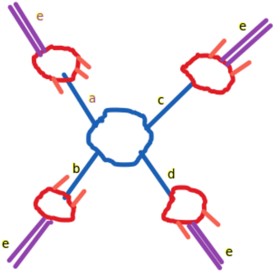
In the collinear approximation we can get them within the NLO QED structure function formalism

- F.A.Berends, W.L. van Neerven, G.J.Burgers, NPB'1988
- A.A., K.Melnikov, PRD'2002; A.A. JHEP'2003

QED NLO master formula

The NLO Bhabha cross section reads

$$d\sigma = \sum_{a,b,c,d=e,\bar{e},\gamma} \int_{\bar{z}_1}^{1} dz_1 \int_{\bar{z}_2}^{1} dz_2 \mathcal{D}^{\text{str}}_{ae}(z_1) \mathcal{D}^{\text{str}}_{b\bar{e}}(z_2)$$
$$\times \left[d\sigma^{(0)}_{ab\to cd}(z_1, z_2) + d\bar{\sigma}^{(1)}_{ab\to cd}(z_1, z_2) \right]$$
$$\times \int_{\bar{y}_1}^{1} \frac{dy_1}{Y_1} \int_{\bar{y}_2}^{1} \frac{dy_2}{Y_2} \mathcal{D}^{\text{frg}}_{ec}\left(\frac{y_1}{Y_1}\right) \mathcal{D}^{\text{frg}}_{\bar{e}d}\left(\frac{y_2}{Y_2}\right)$$
$$+\mathcal{O}\left(\alpha^n L^{n-2}, \frac{m_e^2}{s}\right)$$



$\alpha^2 L^2$ and $\alpha^2 L^1$ terms are completely reproduced [A.A., E.Scherbakova, JETP Lett. 2006; PLB 2008] || $\bar{e} \equiv e^+$

High-order ISR in e^+e^- annihilation (I)

$$\begin{aligned}
\frac{d\sigma_{e^+e^-}}{ds'} = \frac{1}{s}\sigma^{(0)}(s')\Bigg[&\mathcal{D}_{e^+e^+}\left(N,\frac{\mu^2}{m_e^2}\right)\tilde{\sigma}_{e^+e^-}\left(N,\frac{s'}{\mu^2}\right)\mathcal{D}_{e^-e^-}\left(N,\frac{\mu^2}{m_e^2}\right)\\
&+\mathcal{D}_{\gamma e^+}\left(N,\frac{\mu^2}{m_e^2}\right)\tilde{\sigma}_{e^-\gamma}\left(N,\frac{s'}{\mu^2}\right)\mathcal{D}_{e^-e^-}\left(N,\frac{\mu^2}{m_e^2}\right)\\
&+\mathcal{D}_{e^+e^+}\left(N,\frac{\mu^2}{m_e^2}\right)\tilde{\sigma}_{e^+\gamma}\left(N,\frac{s'}{\mu^2}\right)\mathcal{D}_{\gamma e^-}\left(N,\frac{\mu^2}{m_e^2}\right)\\
&+\mathcal{D}_{\gamma e^+}\left(N,\frac{\mu^2}{m_e^2}\right)\tilde{\sigma}_{\gamma\gamma}\left(N,\frac{s'}{\mu^2}\right)\mathcal{D}_{\gamma e^-}\left(N,\frac{\mu^2}{m_e^2}\right)\Bigg]
\end{aligned}$$

J. Ablinger, J. Blümlein, A. De Freitas and K. Schönwald, "Subleading Logarithmic QED Initial State Corrections to $e^+e^- \to \gamma^*/Z^{0*}$ to $O(\alpha^6 L^5)$," NPB 955 (2020) 115045

High-order ISR in e^+e^- annihilation (II)

$$\frac{d\sigma_{e^+e^-\to\gamma^*}}{ds'} = \frac{1}{s}\sigma^{(0)}(s') \sum_{a,b=e^-,\gamma,e^+} D_{ae^-} \otimes \tilde{\sigma}_{ab\to\gamma^*} \otimes D_{be^+}$$

Table. Orders of different contributions:

$a\backslash b$	e^+	γ	e^-
e^-	$D_{e^-e^-}D_{e^+e^+}\sigma_{e^-e^+}$ LO (1)	$D_{\gamma e^-}D_{e^-e^-}\sigma_{e^-\gamma}$ NLO $(\alpha^2 L)$	$D_{e^-e^-}D_{e^-e^+}\sigma_{e^-e^-}$ NNLO $(\alpha^4 L^2)$
γ	$D_{\gamma e^-}D_{e^+e^+}\sigma_{e^+\gamma}$ NLO $(\alpha^2 L)$	$D_{\gamma e^-}D_{\gamma e^+}\sigma_{\gamma\gamma}$ NNLO $(\alpha^4 L^2)$	$D_{\gamma e^-}D_{e^-e^+}\sigma_{e^-\gamma}$ NLO $(\alpha^4 L^3)$
e^+	$D_{e^+e^-}D_{e^+e^+}\sigma_{e^+e^+}$ NNLO $(\alpha^4 L^2)$	$D_{e^+e^-}D_{\gamma e^+}\sigma_{e^+\gamma}$ NLO $(\alpha^4 L^3)$	$D_{e^+e^-}D_{e^-e^+}\sigma_{e^+e^-}$ LO $(\alpha^4 L^4)$

Contributions from $D_{e^-e^+}$ and $D_{e^+e^-}$ are missed. They are relevant starting from $\mathcal{O}(\alpha^4 L^4)$

QED NLO DGLAP evolution equations

$$\mathcal{D}_{ba}\left(x, \frac{\mu_R}{\mu_F}\right) = \delta_{ab}\delta(1-x) + \sum_{c=e,\gamma,\bar{e}} \int_{\mu_R^2}^{\mu_F^2} \frac{dt}{t} \int_x^1 \frac{dy}{y} P_{bc}(y,t)\mathcal{D}_{ca}\left(\frac{x}{y}, \frac{\mu_R}{t}\right)$$

μ_F is a factorization (energy) scale

μ_R is a renormalization (energy) scale

D_{ba} is a parton distribution function (PDF)

P_{bc} is a splitting function or kernel of the DGLAP equation

N.B. In QED $\mu_R = m_e \approx 0$ is the natural choice

Initial conditions

$\mathcal{D}_{ba}^{\text{ini}}$ is the initial approximation in iterative solutions

$$\begin{aligned}
\mathcal{D}_{ee}^{\text{ini}}(x,\mu_R,m_e) &= \delta(1-x) + \frac{\bar{\alpha}(\mu_R)}{2\pi} d_{ee}^{(1)}(x,\mu_R,m_e) + \mathrm{O}(\alpha^2) \\
\mathcal{D}_{\gamma e}^{\text{ini}}(x,\mu_R,m_e) &= \frac{\bar{\alpha}(\mu_R)}{2\pi} P_{\gamma e}^{(0)}(x) + \mathrm{O}(\alpha^2) \\
d_{ee}^{(1)}(x,\mu_R,m_e) &= \left[\frac{1+x^2}{1-x}\left(\ln\frac{\mu_R^2}{m_e^2} - 2\ln(1-x) - 1\right)\right]_+
\end{aligned}$$

They are defined from matching to perturbative calculations, see below

QED splitting functions

The perturbative splitting functions are

$$P_{ba}(x, \bar{\alpha}(t)) = \frac{\bar{\alpha}(t)}{2\pi} P_{ba}^{(0)}(x) + \left(\frac{\bar{\alpha}(t)}{2\pi}\right)^2 P_{ba}^{(1)}(x) + \mathcal{O}(\alpha^3)$$

$$\text{e.g.} \quad P_{ee}^{(0)}(x) = \left[\frac{1+x^2}{1-x}\right]_+$$

They come from direct loop calculations, see, e.g., review "Partons in QCD" by G. Altarelli. For instance, $P_{ba}^{(1)}(z)$ comes from 2-loop calculations.

The splitting functions can be obtained by reduction of the ones known in QCD to the abelian case of QED.

$\bar{\alpha}(t)$ is the QED running coupling constant in the $\overline{\text{MS}}$ scheme

$\mathcal{O}(\alpha)$ matching

The expansion of the master formula for ISR gives

$$d\sigma^{(1)}_{e\bar{e}\to\gamma^*} = \frac{\alpha}{2\pi}\left\{2LP^{(0)} \otimes d\sigma^{(0)}_{e\bar{e}\to\gamma^*} + 2d^{(1)}_{ee} \otimes d\sigma^{(0)}_{e\bar{e}\to\gamma^*}\right\} + d\,\bar{\sigma}^{(1)}_{e\bar{e}\to\gamma^*} + \mathcal{O}\left(\alpha^2\right)$$

We know the massive $d\sigma^{(1)}$ and massless $d\,\bar{\sigma}^{(1)}$ ($m_e \to 0$ with $\overline{\mathrm{MS}}$ subtraction) results in $\mathcal{O}(\alpha)$. E.g.

$$\frac{d\sigma^{(1)}_{e\bar{e}\to\gamma^*}}{d\sigma^{(0)}_{e\bar{e}\to\gamma^*}} \sim \frac{\alpha}{\pi}\left[\frac{1+z^2}{1-z}\right]_+\left(\ln\frac{s}{m_e^2} - 1\right) + \delta(1-z)(\ldots), \quad z \equiv \frac{s'}{s}$$

A scheme dependence comes from here

A factorization scale dependence is also from here

Running coupling constant

Running of α_{QED} is known, e.g., P.Baikov, K.Chetyrkin et al., NPB '2013

$$\bar{\alpha}(t) = \frac{\alpha(\mu_R^2)}{1+\Pi(\mu_R^2/t)}, \qquad \alpha \equiv \alpha(\mu_R^2 = m_e^2) \approx \frac{1}{137.036}$$

$$\Longrightarrow \bar{\alpha}(t) = \alpha\left\{1 + \frac{\alpha}{2\pi}\left(-\frac{10}{9} + \frac{2}{3}L\right) + \left(\frac{\alpha}{2\pi}\right)^2\left(-\frac{1085}{324} \right.\right.$$
$$\left.\left. +4\zeta_3 - \frac{13}{27}L + \frac{4}{9}L^2\right) + \mathcal{O}\left(\alpha^3(\mu_R)\right)\right\}$$

The same $\overline{\text{MS}}$ scheme is used here, $L \equiv \ln(t/\mu_R^2)$

Note that here only electron loops are taken into account. Other contributions can be added.

Iterative solution

The NLO "electron in electron" PDF reads

$$\begin{aligned}
\mathcal{D}_{ee}(x,\mu_F,m_e) &= \delta(1-x) + \frac{\alpha}{2\pi} L P^{(0)}_{ee}(x) + \frac{\alpha}{2\pi} d^{(1)}_{ee}(x,m_e,m_e) \\
&+ \left(\frac{\alpha}{2\pi}\right)^2 L^2 \left(\frac{1}{2} P^{(0)}_{ee} \otimes P^{(0)}_{ee}(x) + \frac{1}{2} P^{(0)}_{ee}(x) + \frac{1}{2} P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e}(x)\right) \\
&+ \left(\frac{\alpha}{2\pi}\right)^2 L \left(P^{(0)}_{e\gamma} \otimes d^{(1)}_{\gamma e}(x,m_e,m_e) + P^{(0)}_{ee} \otimes d^{(1)}_{ee}(x,m_e,m_e) - \frac{10}{9} P^{(0)}_{ee}(x) + P^{(1)}_{ee}(x)\right) \\
&+ \left(\frac{\alpha}{2\pi}\right)^3 L^3 \left(\frac{1}{6} P^{(0)}_{ee} \otimes P^{(0)}_{ee} \otimes P^{(0)}_{ee}(x) + \frac{1}{6} P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma\gamma} \otimes P^{(0)}_{\gamma e}(x) + \ldots\right) \\
&+ \left(\frac{\alpha}{2\pi}\right)^3 L^2 \left(P^{(0)}_{ee} \otimes P^{(1)}_{ee}(x) + P^{(0)}_{ee} \otimes P^{(0)}_{ee} \otimes d^{(1)}_{ee}(x,m_e,m_e) + \frac{1}{3} P^{(1)}_{ee}(x) - \frac{10}{9} P^{(0)}_{ee} \otimes P^{(0)}_{ee}(x) + \ldots\right) \\
&+ \mathcal{O}(\alpha^2 L^0, \alpha^3 L^1)
\end{aligned}$$

The large logarithm $L \equiv \ln \frac{\mu_F^2}{\mu_R^2}$ with factorization scale $\mu_F^2 \sim s$ or $\sim -t$; and renormalization scale $\mu_R = m_e$.

Required convolution integrals are listed in [A.A. hep-ph/0304063]

Convolution

Convolution operation

$$f \otimes g(x) = \int_0^1 dz \int_0^1 dz' \delta(x - zz') f(z) g(z') = \int_x^1 dz f(z) g\left(\frac{x}{z}\right)$$

Plus prescription

$$\int_x^1 dy\, [f(y)]_+ g(y) = \int_0^1 dy f(y) \Big[g(y)\Theta(y - x) - g(1) \Big]$$

Matching in O(α^2)

Complete 2-loop result: Berends et al. 1988; Blümlein et al., 2011

$$\sigma^{(2)}_{e\bar{e}} = \left(\frac{\alpha}{2\pi}\right)^2 L^2 \sigma^{(0)}_{e\bar{e}} \left(P^{(0)}_{\gamma e} \otimes P^{(0)}_{e\gamma} + \frac{2}{3} P^{(0)}_{ee} + 2P^{(0)}_{ee} \otimes P^{(0)}_{ee} \right)$$
$$+ \left(\frac{\alpha}{2\pi}\right)^2 L \sigma^{(0)}_{e\bar{e}} \left(2d^{(1)}_{\gamma e} \otimes P^{(0)}_{e\gamma} + 2P^{(1)}_{ee} - \frac{40}{9} P^{(0)}_{ee} + 4P^{(0)}_{ee} \otimes d^{(1)}_{ee} \right)$$
$$+ \left(\frac{\alpha}{2\pi}\right)^2 L \left(2\sigma^{(0)}_{e\gamma} P^{(0)}_{\gamma e} + \frac{2}{3} \bar{\sigma}^{(1)}_{e\bar{e}} + 2\sigma^{(1)}_{e\bar{e}} \otimes P^{(0)}_{ee} \right)$$
$$+ \mathcal{O}(\alpha^2 L^0) + \mathcal{O}(m_e^2/s)$$

Massification procedure

Two-loop (virtual) corrections tho Bhabha scattering with $m_e \equiv 0$
[Z.Bern, L.J.Dixon, A.Ghinculov, PRD 2001]

Two-loop (virtual+soft) corrections tho Bhabha scattering with $m_e \neq 0$
[A.Penin, PRL 2005; NPB 2006] but for $s, |t|, |u| \gg m_e^2$

Statement: all terms enhanced by large logs $L = \ln(Q^2/m_e^2)$ can be restored

The result of A.Penin was reproduced by adding universal terms to the massless result [T.Becher, K.Melnikov, JHEP 2007]

McMule – NNLO QED Corrections for Low-Energy Experiments
[P.Banerjee, T.Engel, A.Signer, Y.Ulrich, SciPost Phys. 2020]

Applications

Current work:

- ISR in electron-positron annihilation $e^+e^- \to \gamma^*, Z^*$
  "Higher-order NLO initial state radiative corrections to e^+e^- annihilation revisited"
- $\mathcal{O}(\alpha^3L^2)$ corrections to muon decay spectrum: relevant for future experiments on Dirac vs. Majorana neutrino discrimination

Near future plans:

- Implementation into ZFITTER, production of benchmarks, tuned comparisons with KKMC which uses YFS exponentiation for ISR
- Application to different e^+e^- annihilation channels and asymmetries within the SANC project
- $\mathcal{O}(\alpha^3L^2)$ corrections to muon-electron scattering for MUonE experiment

QED PDFs vs. QCD ones

Common properties:

- QED splitting functions = abelian part of QCD ones
- The same structure of DGLAP evolution equations
- The same Drell-Yan-like master formula with factorization
- Factorization scale and scheme dependence

Peculiar properties:

- QED PDFs are calculable
- QED PDFs are less inclusive
- QED renormalization scale $\mu_R = m_e$ is preferable
- QED PDFs can (do) lead to huge corrections
- QED cross-checks QCD

Outlook

- QED NLO PDFs are derived in a consistent way
- Having high theoretical precision for the normalization processes $e^+e^- \to e^+e^-$, $e^+e^- \to \mu^+\mu^-$, and $e^+e^- \to 2\gamma$ is crucial for future e^+e^- colliders, especially for the Tera-Z mode
- There are several two-loop QED results, but leading higher order corrections are also numerically important
- New Monte Carlo codes are required
- Semi-analytic codes are relevant for cross-checks and benchmarks
- Comparisons with recent results of Blümlein et al. show a serious disagreement (even in the leading logs) due to four separate issues
- A bug in QCD NLO PDFs is found (?)
- Our results are relevant for several studies in future experiments

The electron is as inexhaustible as the atom [V. Lenin '1908]