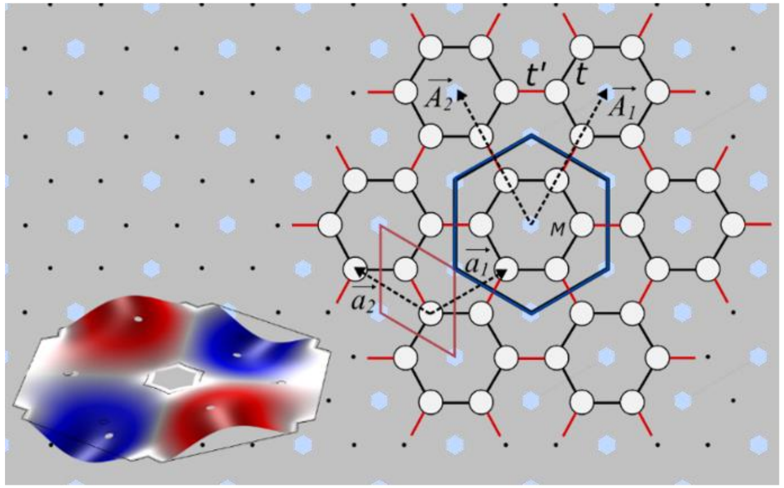
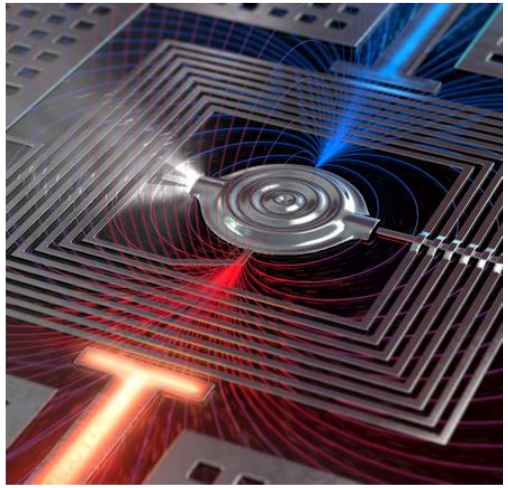
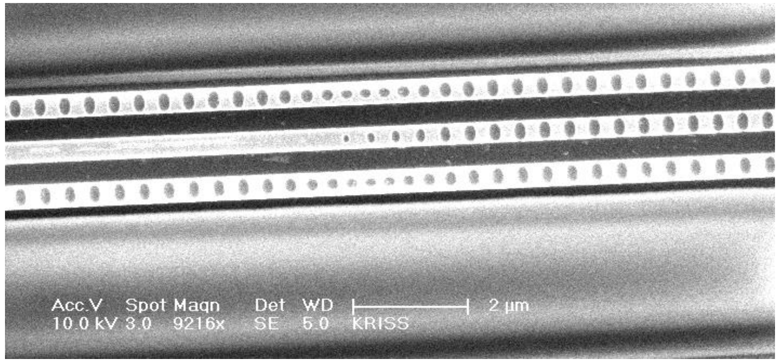
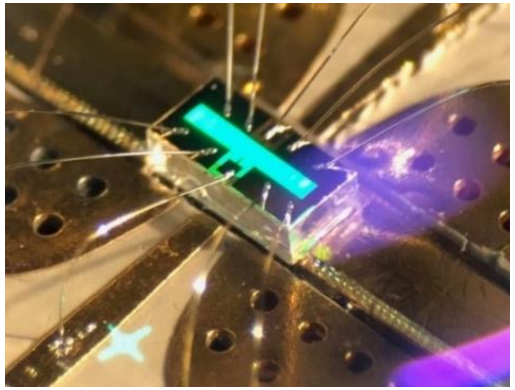
Mechanical Vibrations and Waves for Quantum Technology









May 19, 2023

The 12th School of Mesoscopic Physics: Hybrid Quantum Systems

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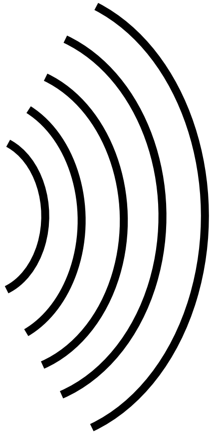
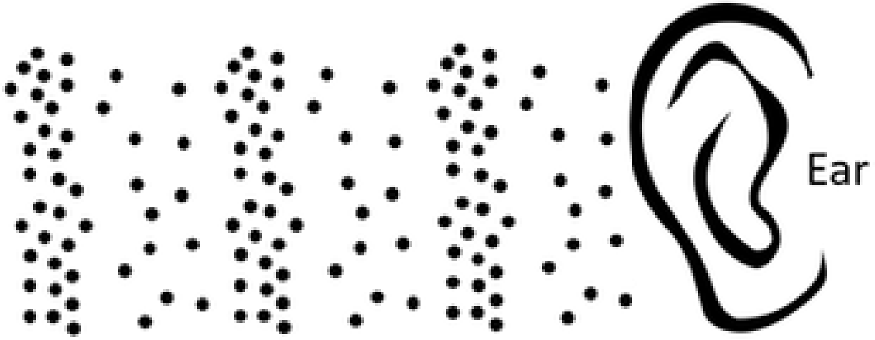
Hybrid Quantum Systems Team
Quantum Technology Institute
Korea Research Institute of Standards and Science

Vibrations and Waves

Musial Instruments | vibrating object | Radiation | Wave Propagation | Sensor





Vibration: **time-periodic** motion of a mechanical object

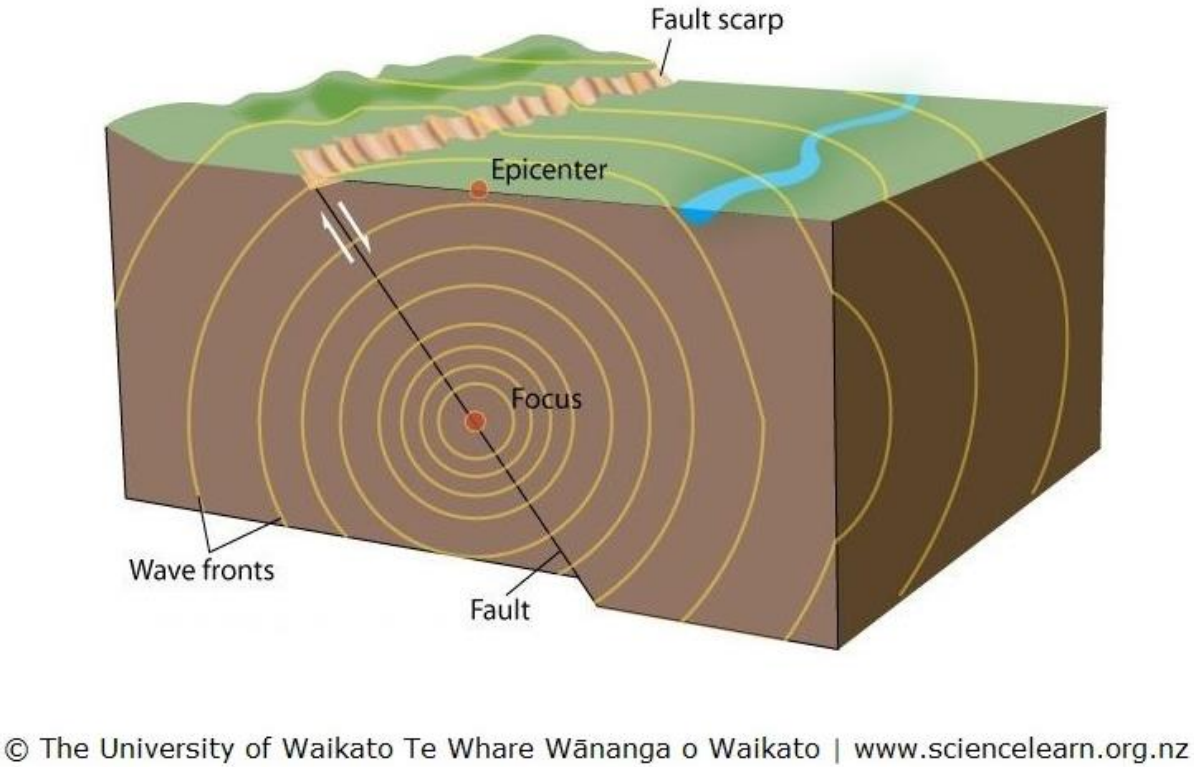
Wave: propagation of energy and information via **a space**

Relevant Physical Parameters:

- Density or mass
- Tension or internal stress
- Elastic properties
- Physical dimensions

Vibrations and Waves: that we want to avoid..

Tacoma Bridge



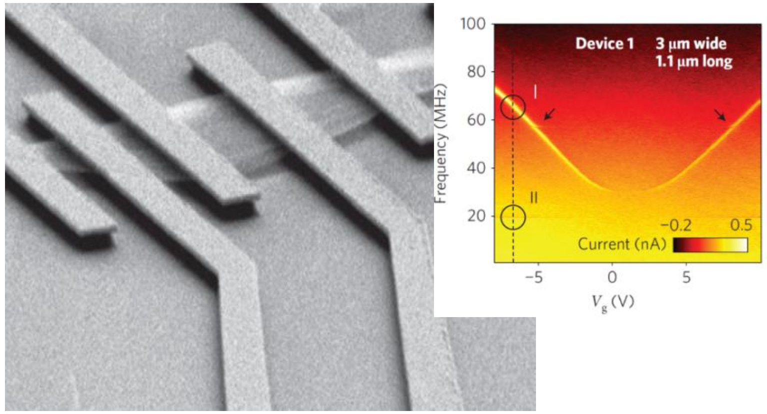
Seismic Waves

We might want to remove vibrations and waves at the macroscale..

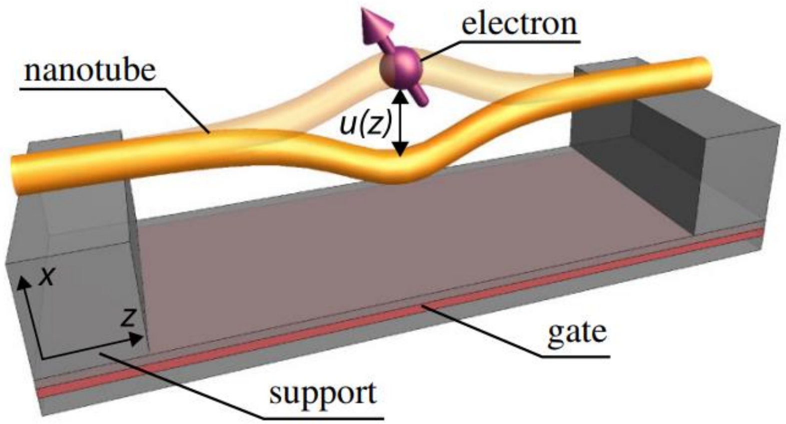
Vibrations and Waves: for fun

We can take advantage of waves, but be careful about sharks !!!

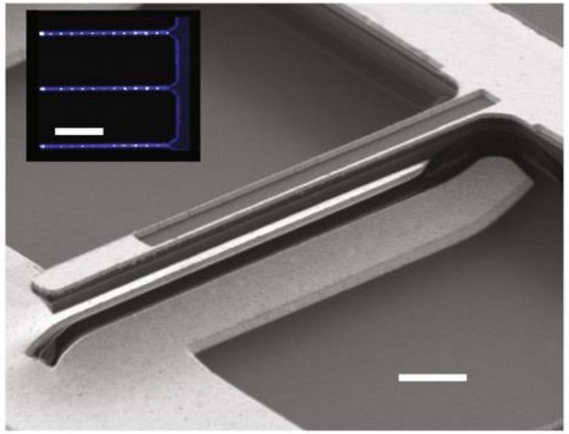
Vibrations and Waves at the Nanoscale



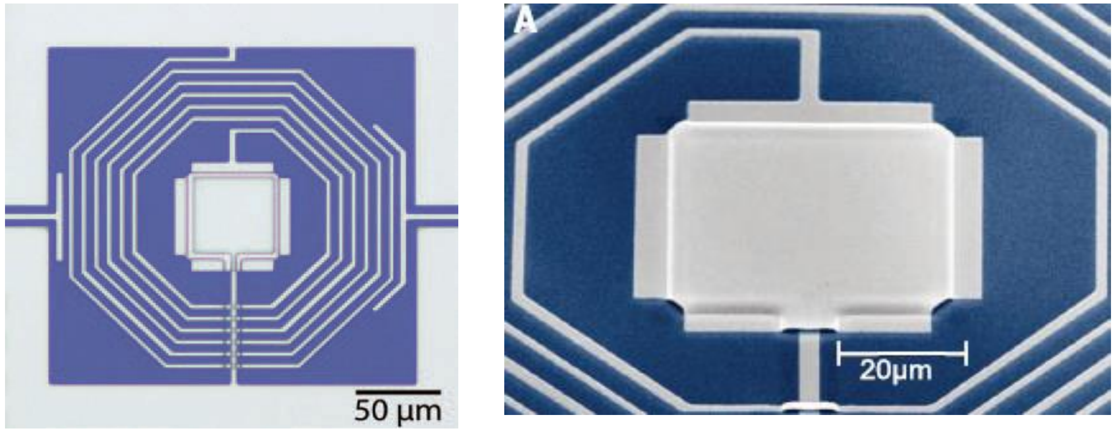
Nat. Nanotechnol. **4**, 861-867 (2009)



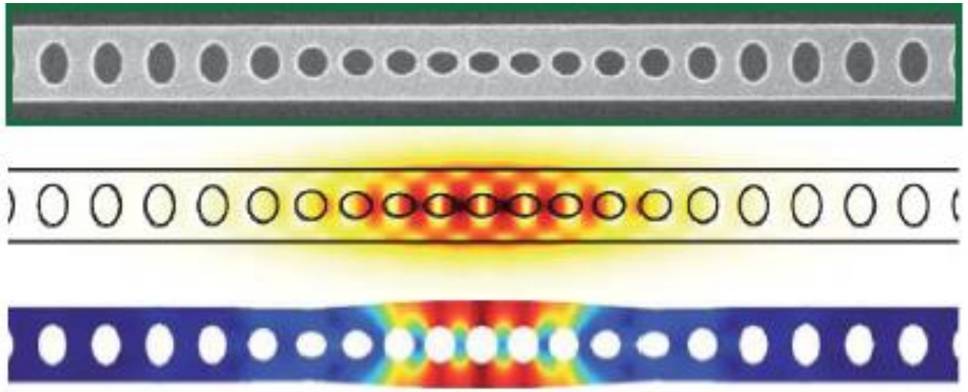
Phys. Rev. Lett, **9**, 2012 (2018)



Nat. Commun, **9**, 2012 (2018)



Science *349*, 952 – 955 (2015)



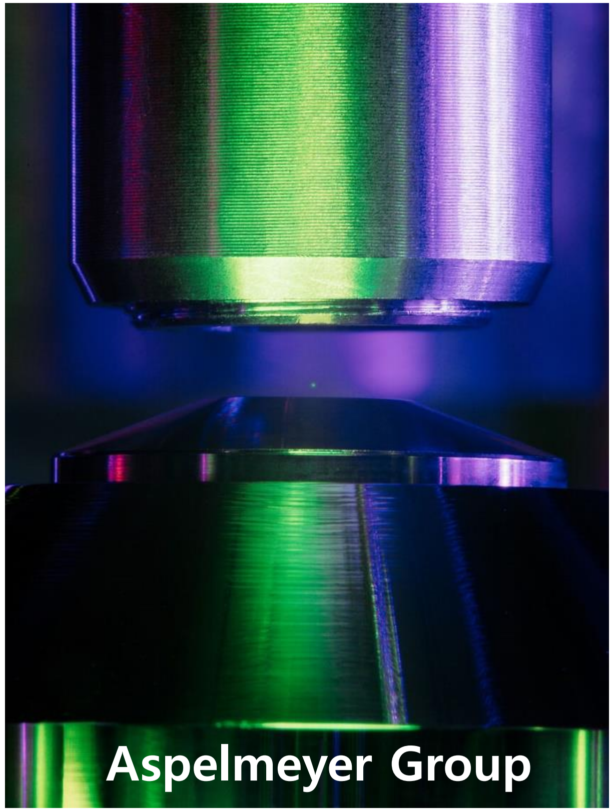
Nature *472*, 69 – 73 (2011)

Nanomechanical Systems and Their Merits

- small device footprint ($\lambda_{phonons} \ll \lambda_{photons}$)
- high-frequency operations
- low-energy loss (high Q)
- electromechanical and optomechanical coupling
- coupling with qubits, spins, charges, etc.

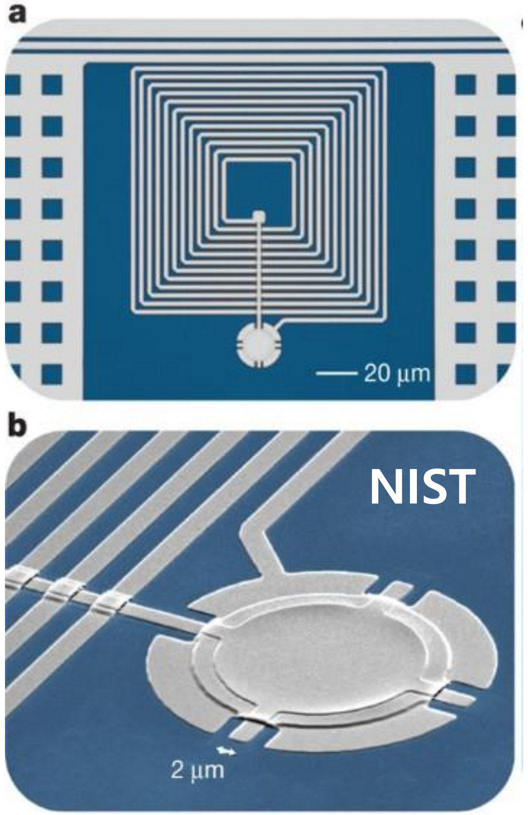
Nanomechanical Systems are perfect platforms for interconnecting different physical systems!

Mechanical Vibrations and Waves for Quantum Technology



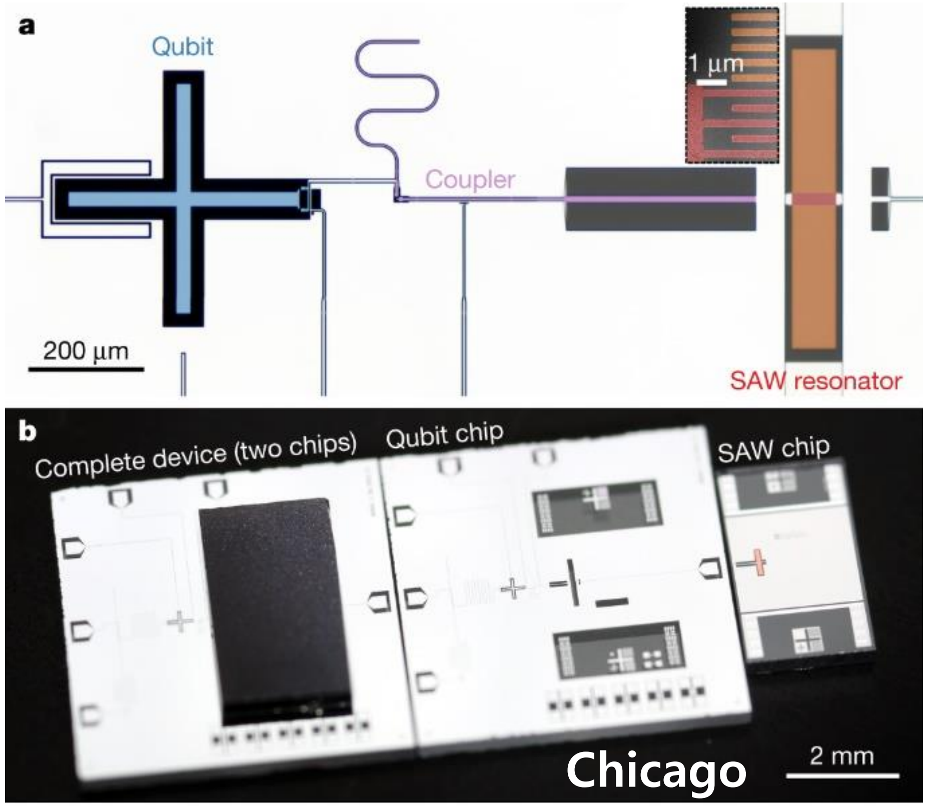
levitated nanoparticles in optical traps

Motional quantum ground state: *Science*, **367**, 892-895 (2020)



microwave circuit optomechanics

Ground state cooling: J.D. Teufel, *et al. Nature* **471**, 204-208(2011)



Superconducting qubit and surface acoustic wave resonator

Qubit-SAW phonon coupling: *Nature* **563**, 661-665 (2018)

Mechanical Vibrations of a Single Body

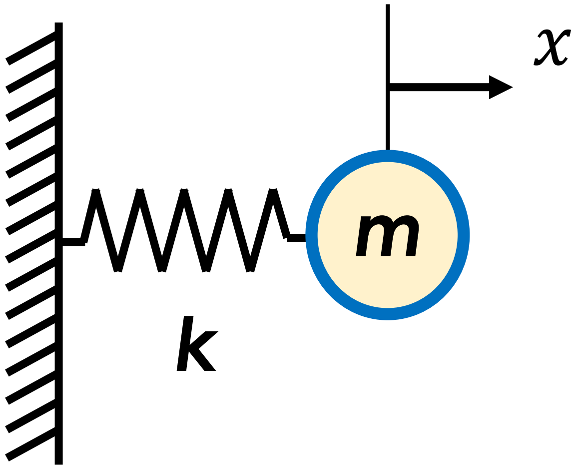
Newton's second law $$m\frac{d^2x}{dt^2} = \sum F$$

$$m\frac{d^2x}{dt^2} = -kx$$

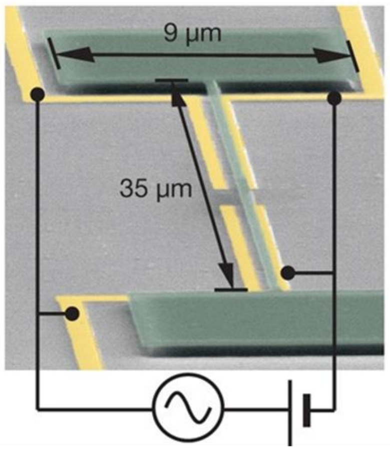
$$m\frac{d^2x}{dt^2} + kx = 0$$

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

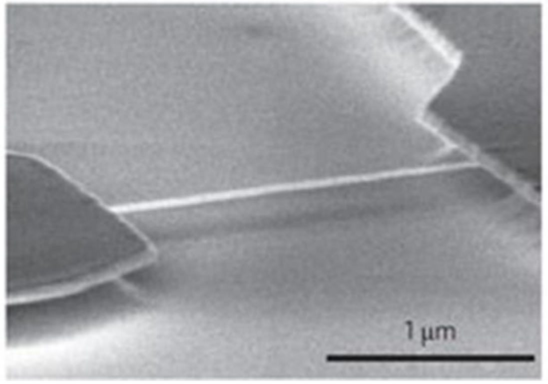
natural frequency $$\omega_0 = \sqrt{\frac{k}{m}}$$



Single degree of freedom



Nature **458**, 1001-1004 (2009)



Nat. Nanotechnol. **4**, 861-867 (2009)

Mechanical Vibrations of a Single Body with Damping

Newton's second law $m\frac{d^2x}{dt^2} = \sum F$

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} + F_0 \sin(\omega t)$$

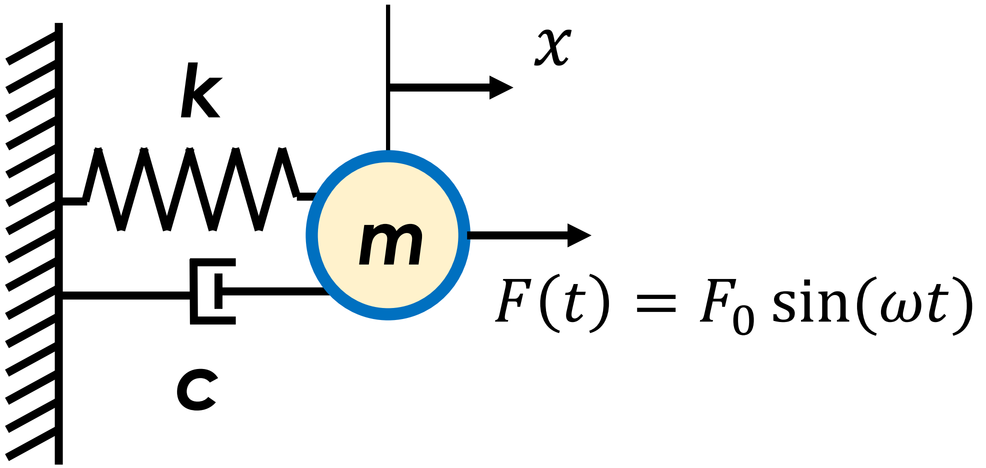
$$\longrightarrow \quad m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_0 \sin(\omega t)$$

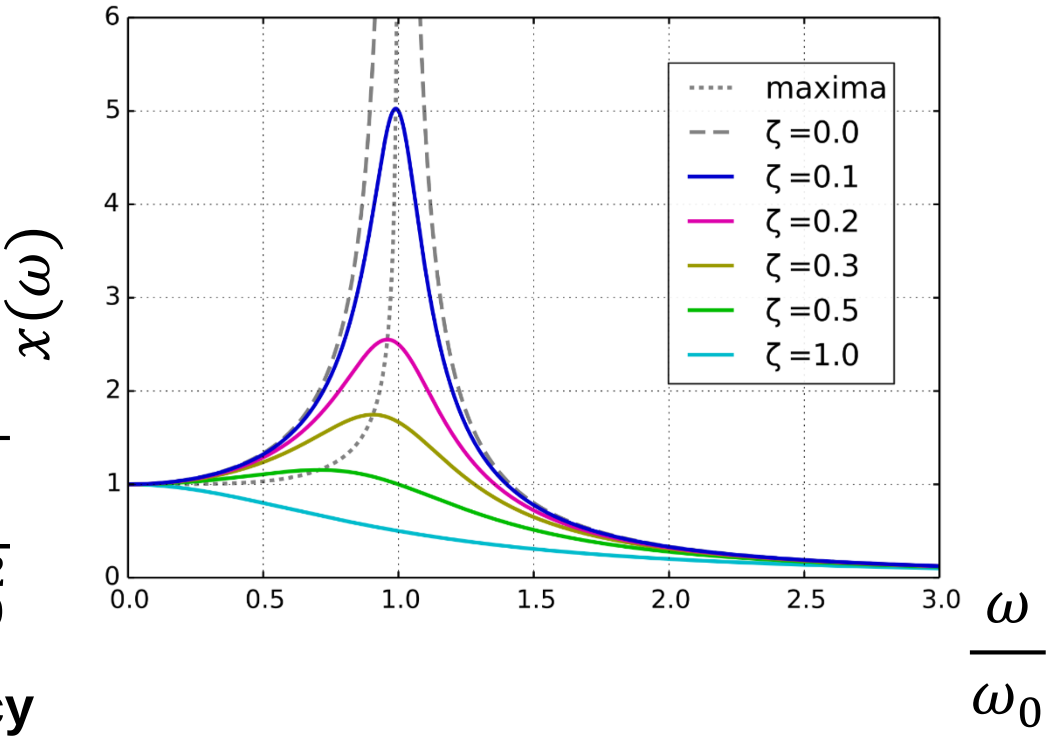
$$\longrightarrow \quad \frac{d^2x}{dt^2} + \Gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\sin(\omega t)$$

$\Gamma = \frac{c}{m}$ **damping rate**

$\omega_0 = \sqrt{\frac{k}{m}}$ **natural frequency**

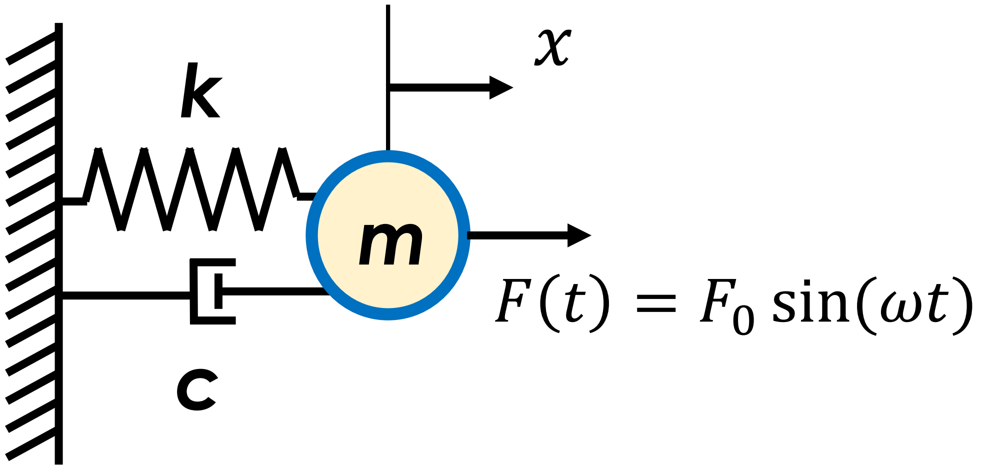
$\omega_r = \omega_0 \sqrt{1 - \frac{\Gamma^2}{2\omega_0^2}}$ **resonance frequency**





Mechanical Vibrations of a Single Body with Damping

$$\frac{d^2x}{dt^2} + \Gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\sin(\omega t)$$



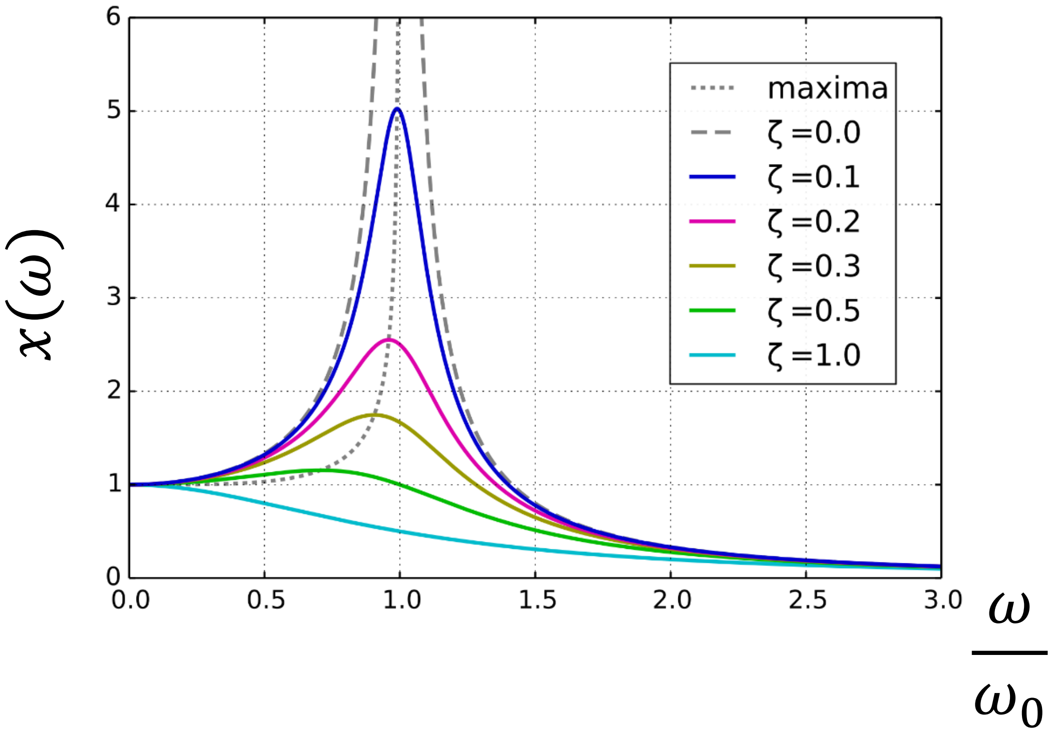
If we calculate the steady state solution with

$$x = x(\omega)\sin(\omega t + \phi)$$

we obtain

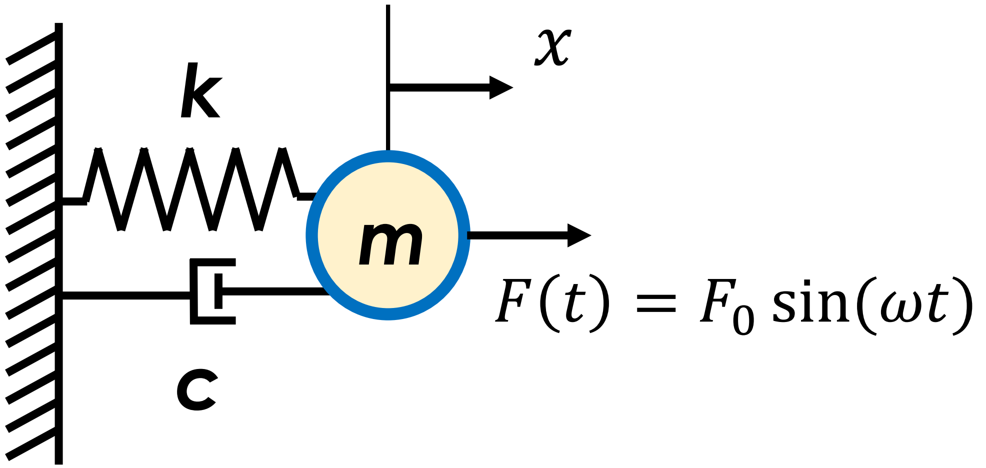
$$x(\omega) = \frac{F_0}{m\sqrt{(\omega\Gamma)^2 + (\omega_0^2 - \omega^2)^2}}$$

$$\phi = \tan^{-1}\left(\frac{\omega\Gamma}{\omega^2 - \omega_0^2}\right)$$



Note:
- a single mass system has one resonant frequency
- the amplitude goes maximum at the resonance
- low damping leads to large displacement

Mechanical Vibrations of a Single Body with Damping



When we analyze resonance data from experiments, **we fit to the Lorentzian curve to obtain the resonance frequency and the damping rate**. How then the displacement response is related to the Lorentzian?

Let's begin with the displacement spectrum we obtained previously.

$$x(\omega) = \frac{F_0}{m\sqrt{(\omega\Gamma)^2 + (\omega_0^2 - \omega^2)^2}}$$

Around the resonance $\omega \approx \omega_0$, we can use the following approximation and insert this to the displacement equation.

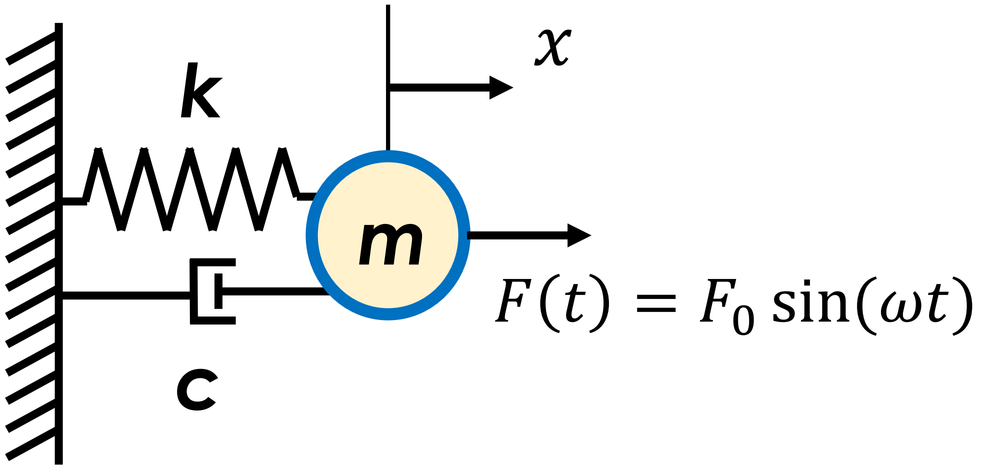
$$(\omega_0^2 - \omega^2) \approx (\omega_0 + \omega)(\omega_0 - \omega) = 2\omega_0(\omega_0 - \omega)$$

The displacement function then becomes

$$x(\omega) = \frac{F_0}{m\sqrt{(\omega_0\Gamma)^2 + \left(2\omega_0(\omega_0 - \omega)\right)^2}} = \frac{F_0}{2m\omega_0\sqrt{\left(\frac{\Gamma}{2}\right)^2 + (\omega_0 - \omega)^2}}$$

Mechanical Vibrations of a Single Body with Damping

In experiment, the response of a mechanical resonator is related to its energy, we should consider the square of the displacement. Thus,

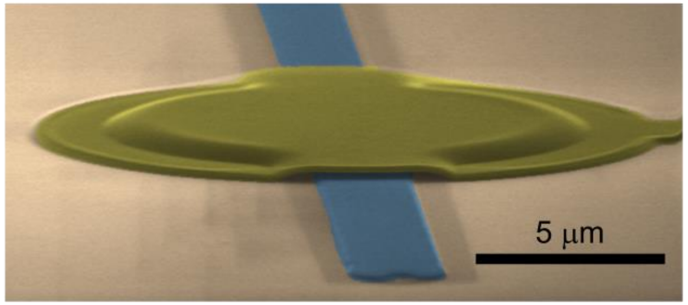


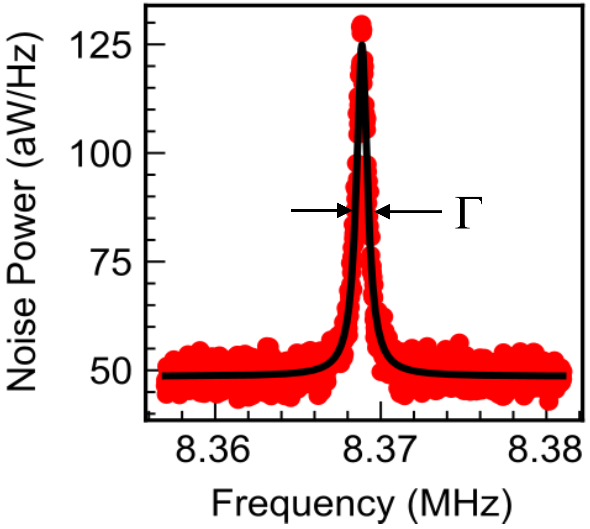
$$|x(\omega)|^2 = \frac{F_0^2}{4m^2\omega_0^2\left[\left(\frac{\Gamma}{2}\right)^2 + (\omega_0 - \omega)^2\right]}$$

$$\propto \frac{1}{\left[\left(\frac{\Gamma}{2}\right)^2 + (\omega_0 - \omega)^2\right]}$$

⟵ This formula is the well known Lorentzian function!

Example) a resonance curve of a nanomechanical resonator





$\omega_0 \approx 2\pi \times 8.369$ MHz

$\Gamma \approx 2\pi \times 810$ Hz

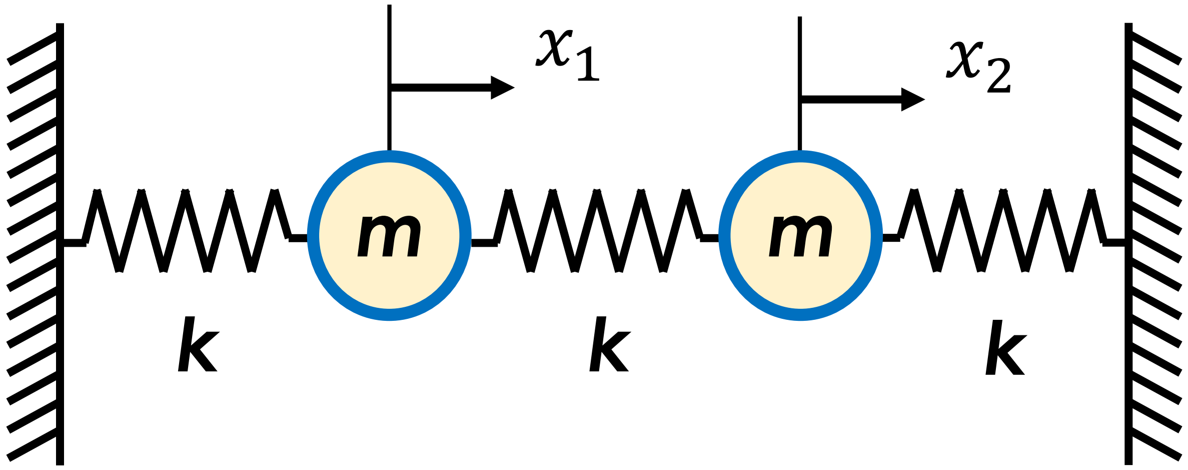
$Q = \frac{\omega_0}{\Gamma}$ **quality factor**

Mechanical Vibrations of a Two-Body System

Let's consider a case where two masses exist. The Newton's second law leads

$$m\frac{d^2x_1}{dt^2} = k(x_2 - x_1) - kx_1$$

$$m\frac{d^2x_2}{dt^2} = k(x_1 - x_2) - kx_2$$



two degrees of freedom

If we express the equations of motion in a matrix form, we get

$$m\frac{d^2}{dt^2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = k\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \longrightarrow m\frac{d^2}{dt^2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} - k\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = 0$$

To calculate the natural frequencies (eigenvalues) of this system, we let $\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}u_1\\ u_2\end{bmatrix}e^{-i\omega t}$ and insert this solution to the above equation, we obtain

$$\begin{bmatrix}-m\omega^2 + 2k & -k\\ -k & -m\omega^2 + 2k\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} = 0 \longrightarrow \begin{vmatrix}-m\omega^2 + 2k & -k\\ -k & -m\omega^2 + 2k\end{vmatrix} = 0$$

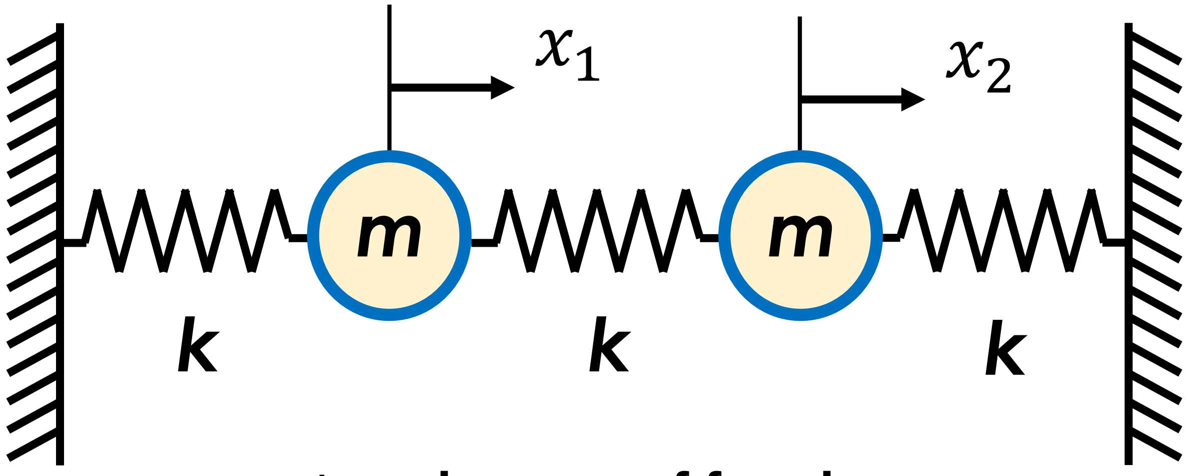
Mechanical Vibrations of a Two-Body System

If we calculate the determinant, we obtain the following characteristic equation:

$$(m\omega^2 - 2k)^2 - k^2 = 0$$

The eigenfrequencies are then given by

$$\omega_{\pm} = \sqrt{\frac{2k \pm k}{m}}$$



two degrees of freedom

If we insert this to the eigenvalue equation, we obtain the following eigenvalue-eigenvector pairs.

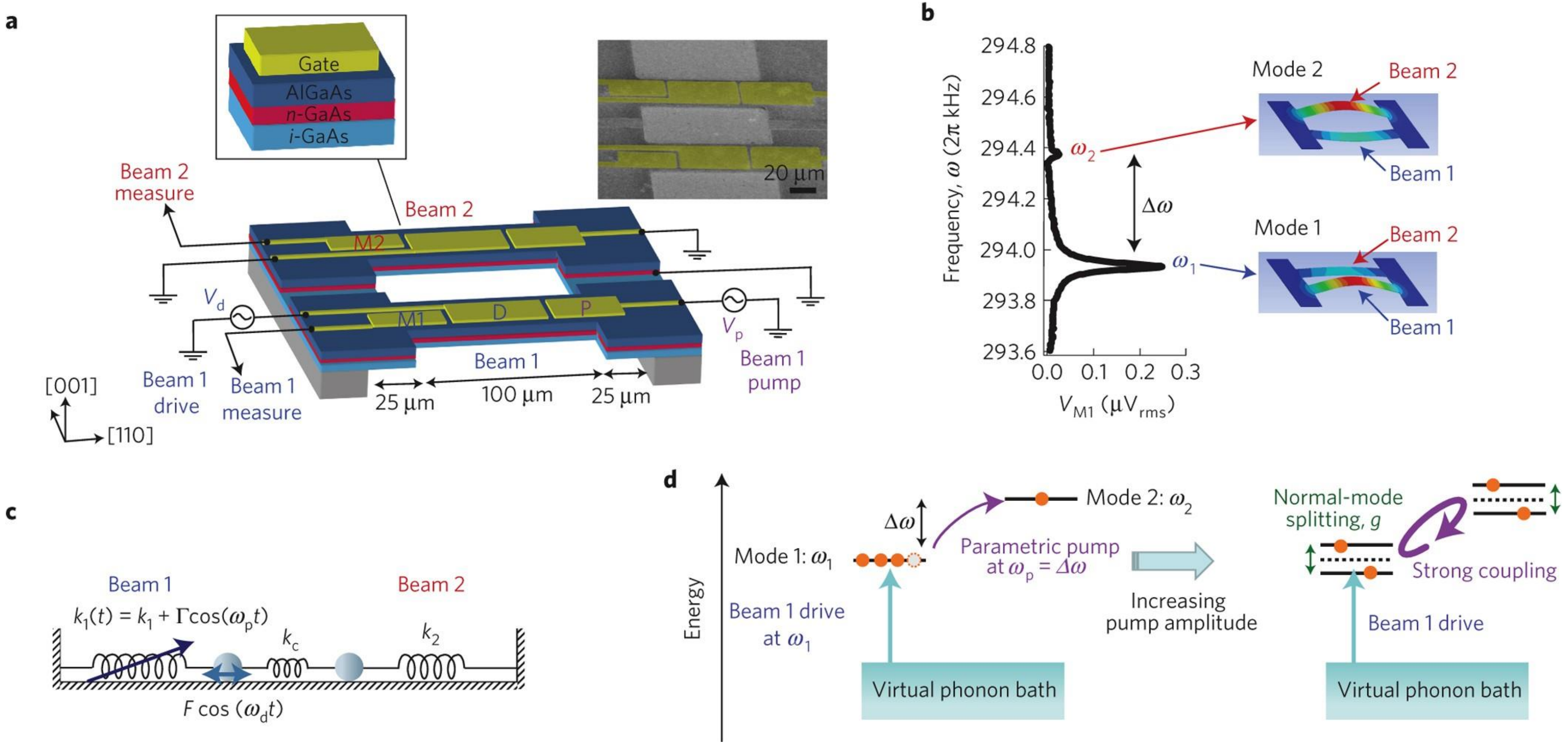
$$\omega_- = \sqrt{\frac{k}{m}} \quad , \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

symmetric eigenmode for the smaller eigenfrequency

$$\omega_+ = \sqrt{\frac{3k}{m}} \quad , \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

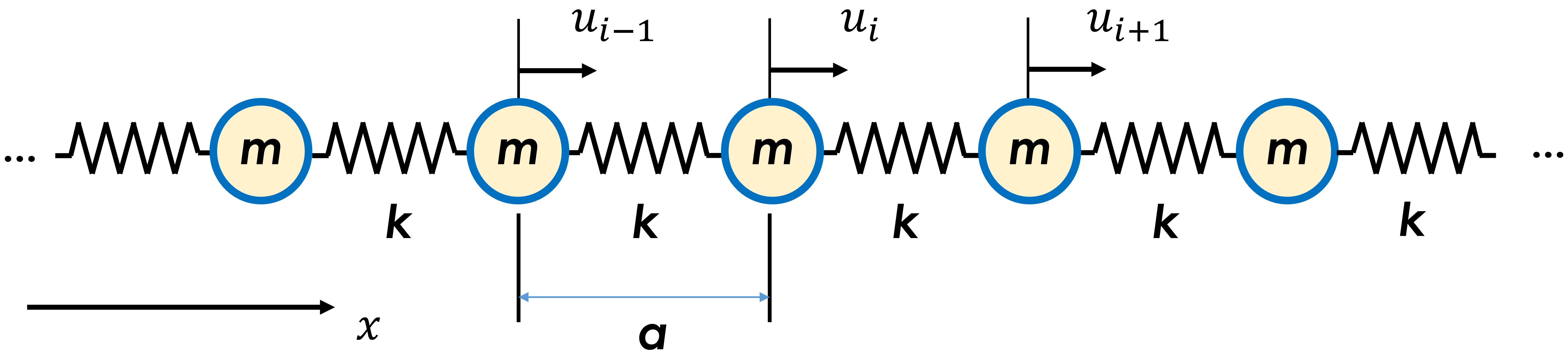
asymmetric eigenmode for the larger eigenfrequency

Research Example of Mechanical Vibrations of a Two-Body System



Coherent phonon manipulation in coupled mechanical resonators, *Nature Physics* **9**, 480-484 (2013)

Mechanical Vibrations of a Many-Body System: Wave



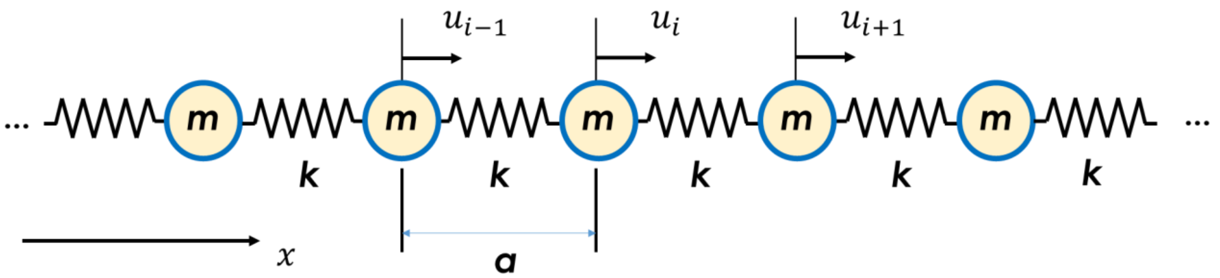
What about if we have **an one-dimensional, infinite array of mass-spring systems**? Here, note that we introduce a new parameter *a* which is the lattice periodicity (the size of a unit cell). Let's write down the equation of motion for this system. In this case, since the periodic nature of this system, we only need to consider the dynamics of a mass in a unit cell. The equation reads

$$m\frac{d^2u_i}{dt^2} = k(u_{i+1} - u_i) + k(u_{i-1} - u_i)$$

Mechanical Vibrations of a Many-Body System: Wave

What this equation implies for with the spatial information?

Let's do some approximation to the equation.



$$u_i = u(x,t) \qquad u_{i\pm1} = u(x \pm a, t)$$

If we insert this to the equation of motion, we obtain.

$$m\frac{\partial^2 u(x,t)}{\partial t^2} = k[u(x+a,t) - u(x,t)] + k[u(x-a,t) - u(x,t)]$$

If we describe $u(x \pm a, t)$ using the Taylor expansion up to the second order with respect to *x*, we get

$$u(x \pm a, t) = u(x,t) \pm a\frac{\partial u(x,t)}{\partial x} + \frac{a^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2}$$

If we insert this expression to the equation, we indeed obtain the famous wave equation!!

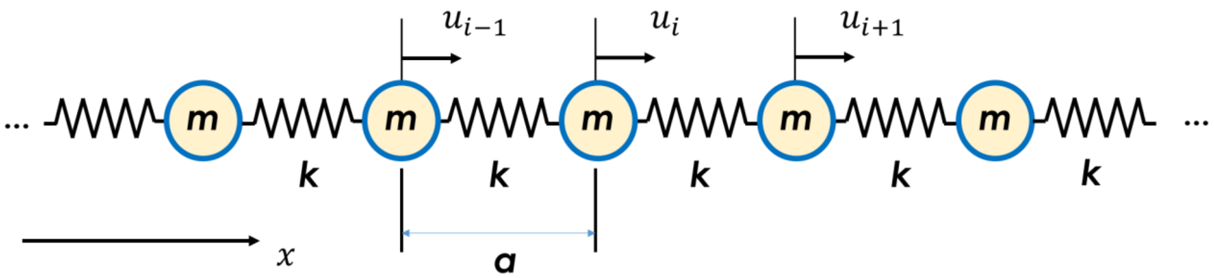
$$m\frac{\partial^2 u(x,t)}{\partial t^2} = ka^2\frac{\partial^2 u(x,t)}{\partial x^2} \longrightarrow \frac{\partial^2 u(x,t)}{\partial t^2} = c^2\frac{\partial^2 u(x,t)}{\partial x^2}$$

wave velocity!

$$c^2 = \frac{k}{m}a^2$$

Mechanical Vibrations of a Many-Body System: Wave

If we insert a plane-wave solution to the wave equation,



$$u(x,t) = u_0 e^{j(qx-\omega t)}$$

We obtain the following (dispersion) relation for the angular frequency ω and the wave vector q

$$\omega = cq$$

So we now understand that this infinite array of masses can support propagating wave which can be described by the wave equation with linear dispersion relation.

However, we also know that we did some (continuum) approximation and the equation does not fully capture the behavior of this system with discrete nature. To see the discrete effects, let's consider the equation of motion again.

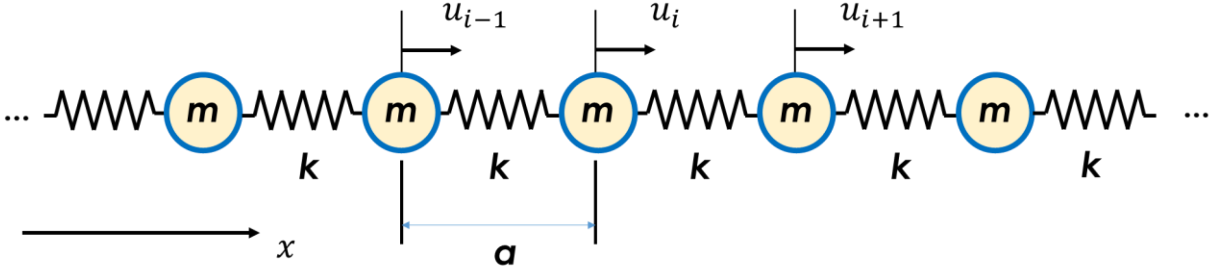
$$m\frac{d^2u_i}{dt^2} = k(u_{i+1} - u_i) + k(u_{i-1} - u_i)$$

Bloch theorem!

Here, we can use a form of the solution, $u_i = u_0 e^{j[q(ia)-\omega t]}$, which leads to $u_{i\pm 1} = u_i e^{\pm jqa}$.
Inserting these to the equation of motion, we get

Mechanical Vibrations of a Many-Body System: Wave

$$m\frac{d^2u_i}{dt^2} = k(u_{i+1} - u_i) + k(u_{i-1} - u_i)$$



$$-m\omega^2 = k\left(e^{jqa} - 1\right) + k\left(e^{-jqa} - 1\right)$$

$$m\omega^2 + k\left(e^{jqa} + e^{-jqa} - 2\right) = m\omega^2 + k(2\cos qa - 2) = m\omega^2 - 4k\sin^2\frac{qa}{2} = 0$$

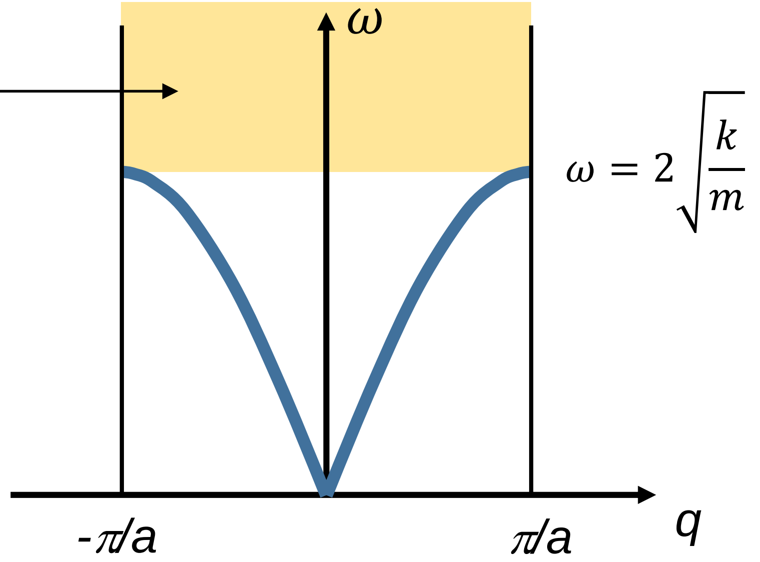
In the end, we obtain the dispersion relation for this one-dimensional monatomic system as

$$\omega^2 = \frac{4k}{m}\sin^2\frac{qa}{2}$$

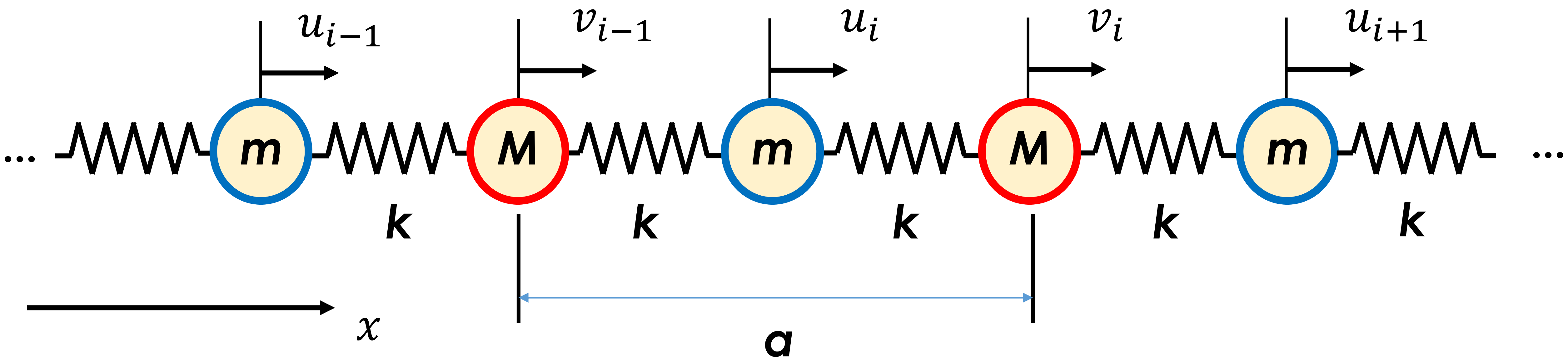
$$\omega^2 \approx \frac{4k}{m}\left(\frac{qa}{2}\right)^2 = \frac{k}{m}a^2q^2 = c^2q^2$$

Long wave approximation, when *qa<<1*

forbidden frequency region



Waves in a One-Dimensional Lattice with Two Bodies in a Unit Cell



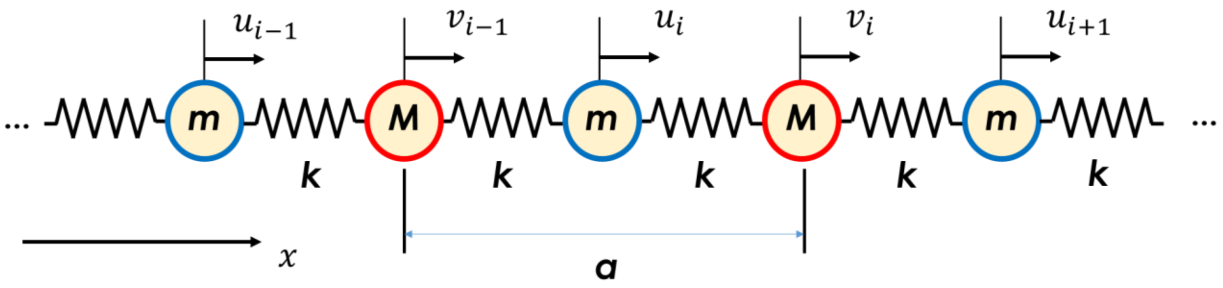
What about if we have **two masses in a unit cell** of an one-dimensional, infinite array of mass-spring systems?

Let's write down the equations of motion!

$$m\frac{d^2u_i}{dt^2} = k(v_i - u_i) + k(v_{i-1} - u_i)$$

$$M\frac{d^2v_i}{dt^2} = k(u_{i+1} - v_i) + k(u_i - v_i)$$

Waves in a One-Dimensional Lattice with Two Bodies in a Unit Cell



Here, the solutions we will use are

$$u_i = u_0 e^{j[q(ia)-\omega t]} \qquad v_i = v_0 e^{j[q(ia)-\omega t]}$$

$$u_{i\pm 1} = u_i e^{\pm jqa} \qquad v_{i\pm 1} = v_i e^{\pm jqa}$$

If we insert these solutions to the equations of motion, we obtain

$$-\omega^2 m u_0 = k(v_0 - u_0) + k\left(v_0 e^{-jqa} - u_0\right)$$

$$-\omega^2 M v_0 = k\left(u_0 e^{jqa} - v_0\right) + k(u_0 - v_0)$$

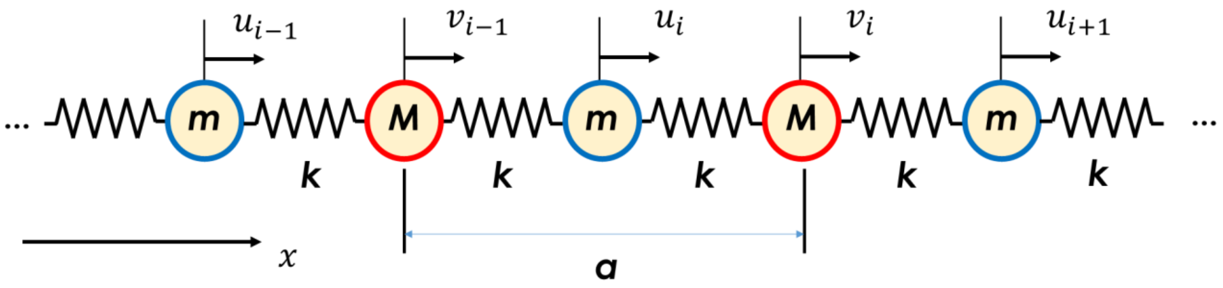
In a matrix form, we can express the equations as

$$\begin{bmatrix} -\omega^2 m & 0 \\ 0 & -\omega^2 M \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} -2k & k\left(1 + e^{-jqa}\right) \\ k\left(1 + e^{jqa}\right) & -2k \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$$

We can easily notice that because the characteristic equation is a 2 x 2 matrix equation, we will have two dispersion curves.

Waves in a One-Dimensional Lattice with Two Bodies in a Unit Cell

If we obtain eigenfrequencies and eigenvectors by solving the characteristic equation,



$$\begin{bmatrix} \omega^2 m - 2k & k\left(1 + e^{-jqa}\right) \\ k\left(1 + e^{jqa}\right) & \omega^2 M - 2k \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = 0$$

If we insert these solutions to the equations of motion, we obtain

$$(\omega^2 m - 2k)(\omega^2 M - 2k) - k^2\left(1 + e^{-jqa}\right)\left(1 + e^{jqa}\right) = 0$$

$$mM\omega^4 - 2k(m + M)\omega^2 + 2k^2(1 - \cos qa) = 0$$

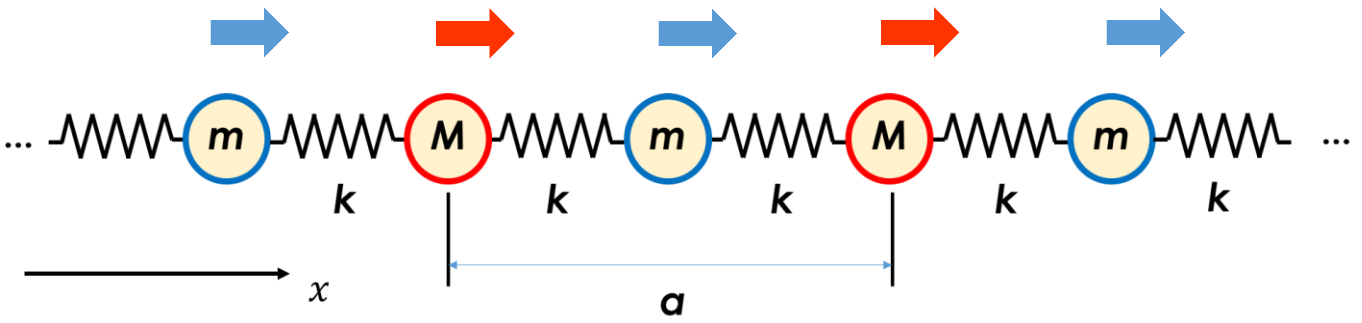
$$\omega^4 - 2k\left(\frac{1}{M} + \frac{1}{m}\right)\omega^2 + \frac{4k^2}{mM}\sin^2\frac{qa}{2} = 0$$

$$\omega_\pm^2 = k\left(\frac{1}{M} + \frac{1}{m}\right) \pm k\sqrt{\left(\frac{1}{M} + \frac{1}{m}\right)^2 - \frac{4}{mM}\sin^2\frac{qa}{2}}$$

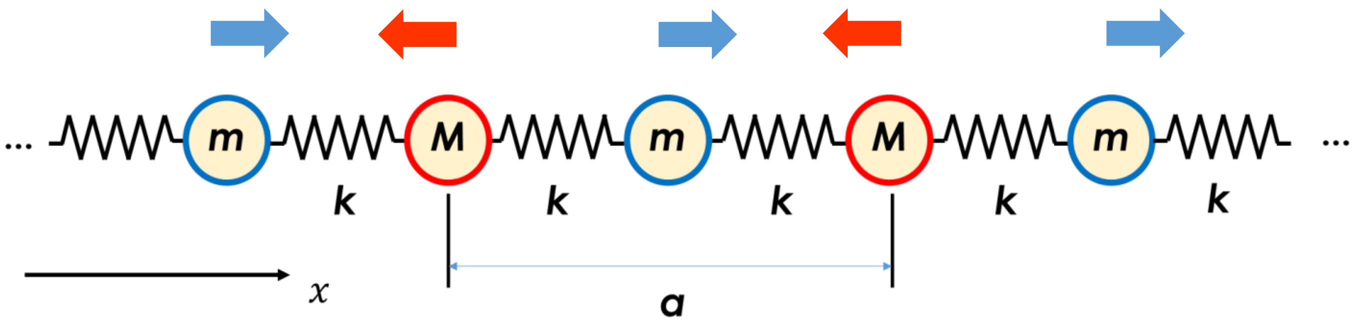
Waves in a One-Dimensional Lattice with Two Bodies in a Unit Cell

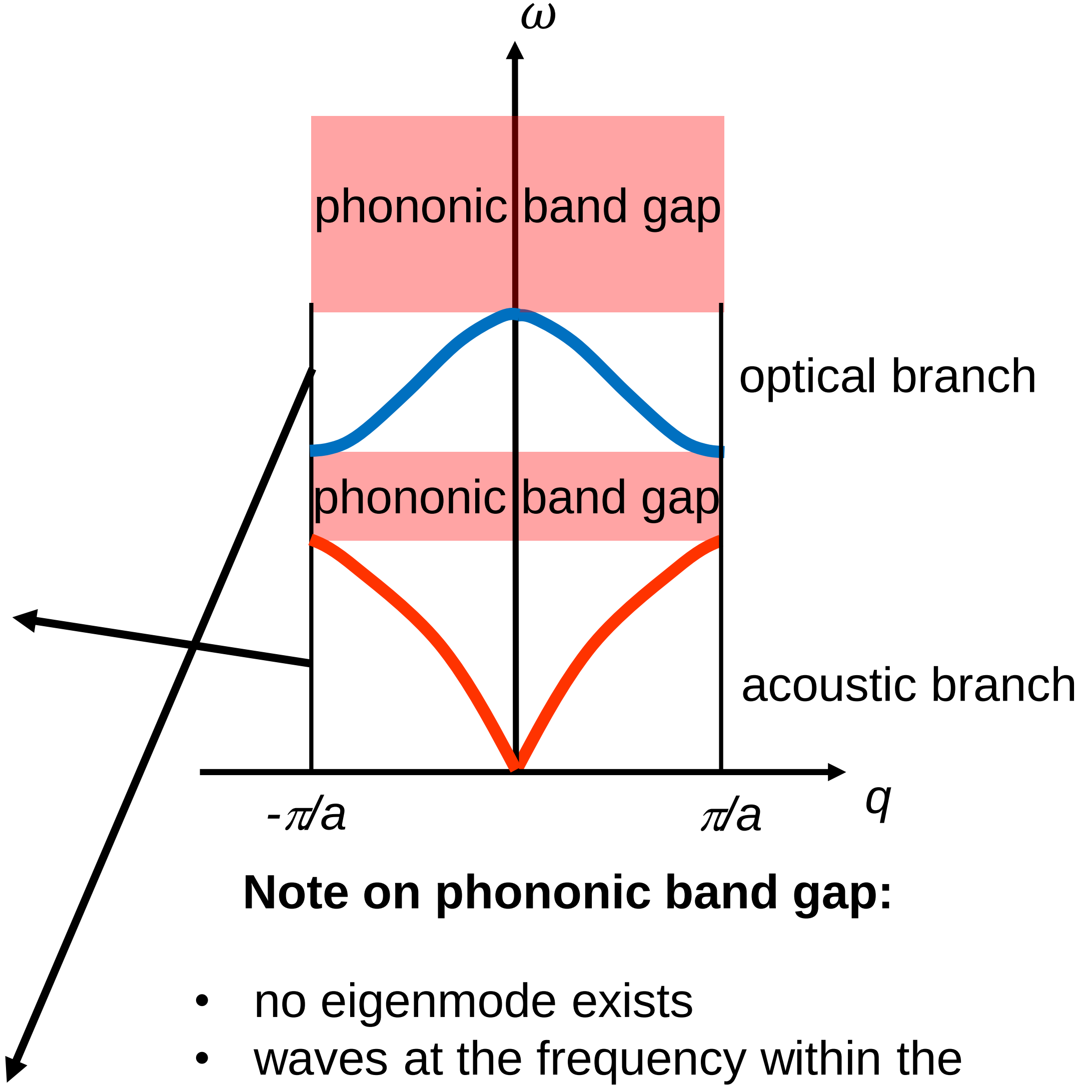
$$\omega_{\pm}^2 = k\left(\frac{1}{M}+\frac{1}{m}\right) \pm k\sqrt{\left(\frac{1}{M}+\frac{1}{m}\right)^2 - \frac{4}{mM}\sin^2\frac{qa}{2}}$$

Eigenvectors for acoustic branch (symmetric)



Eigenvectors for optical branch (asymmetric)

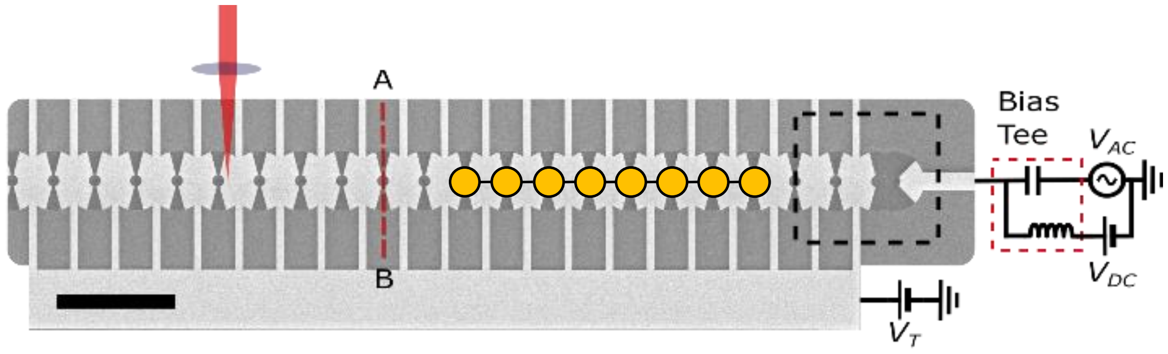




Note on phononic band gap:

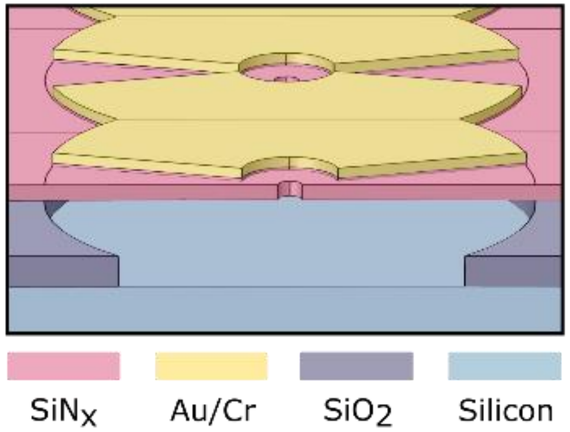
- no eigenmode exists
- waves at the frequency within the band gaps cannot propagate inside the system, but reflected.

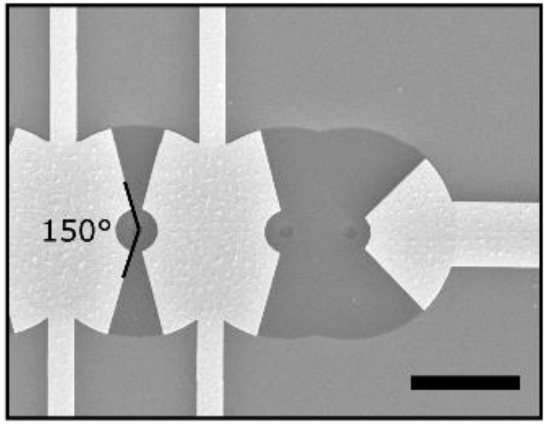
Research Example of Waves in a One-Dimensional Lattice



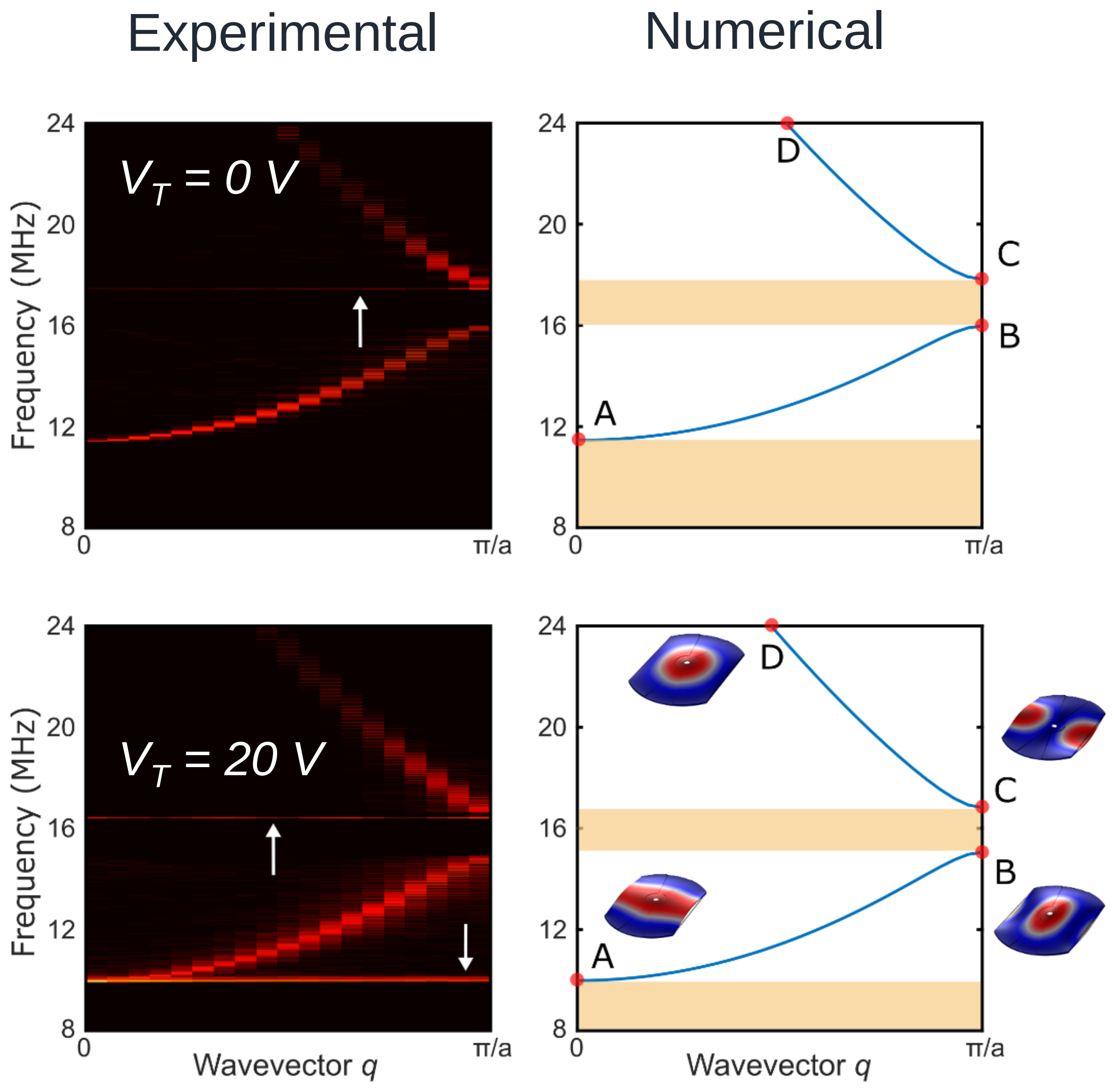
- V_T : Tuning voltage
- V_{AC} : AC excitation voltage
- V_{DC}: DC voltage

Number of unit cells: 120



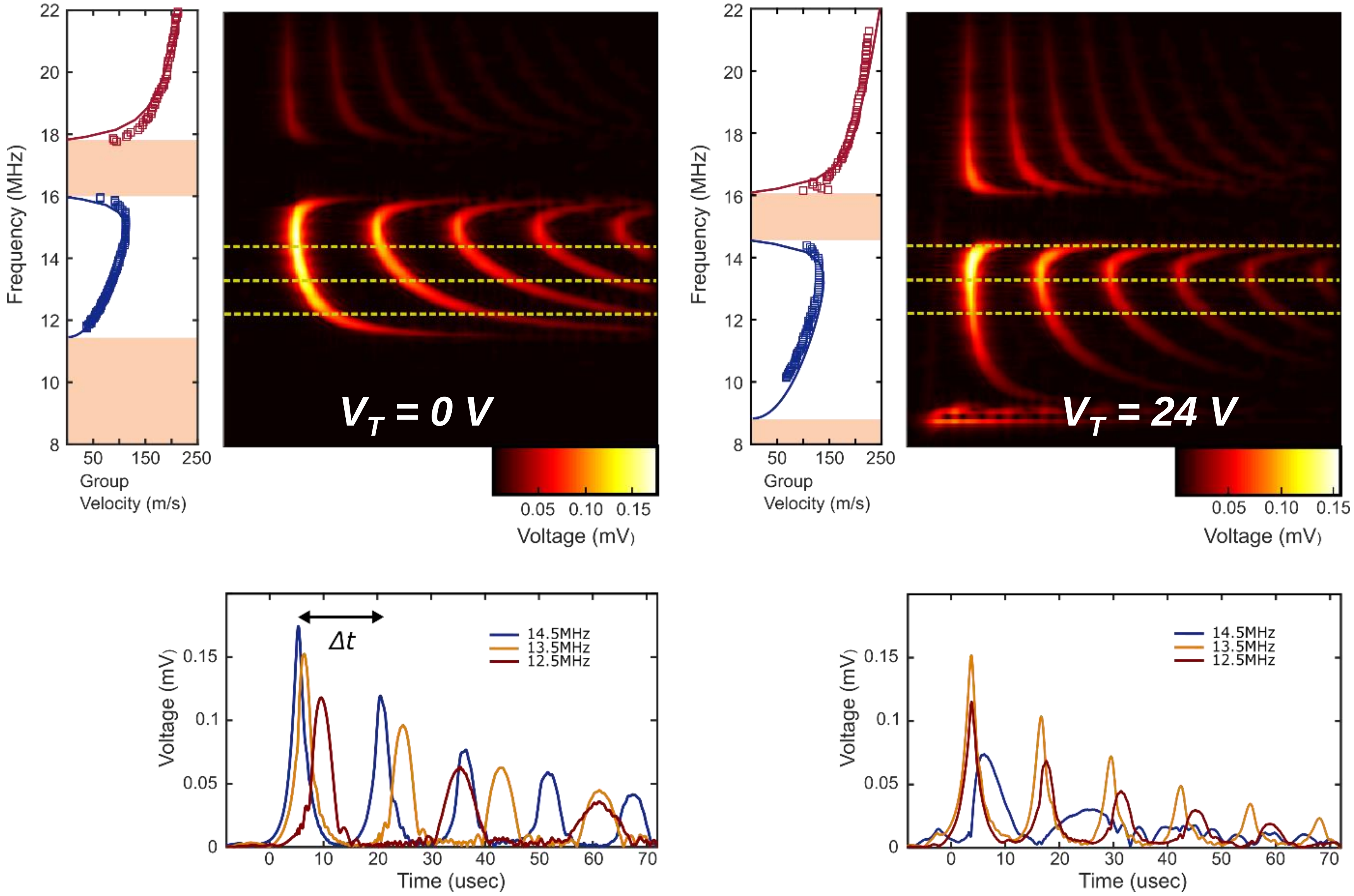


Isotropic HF etching of thermal SiO_2



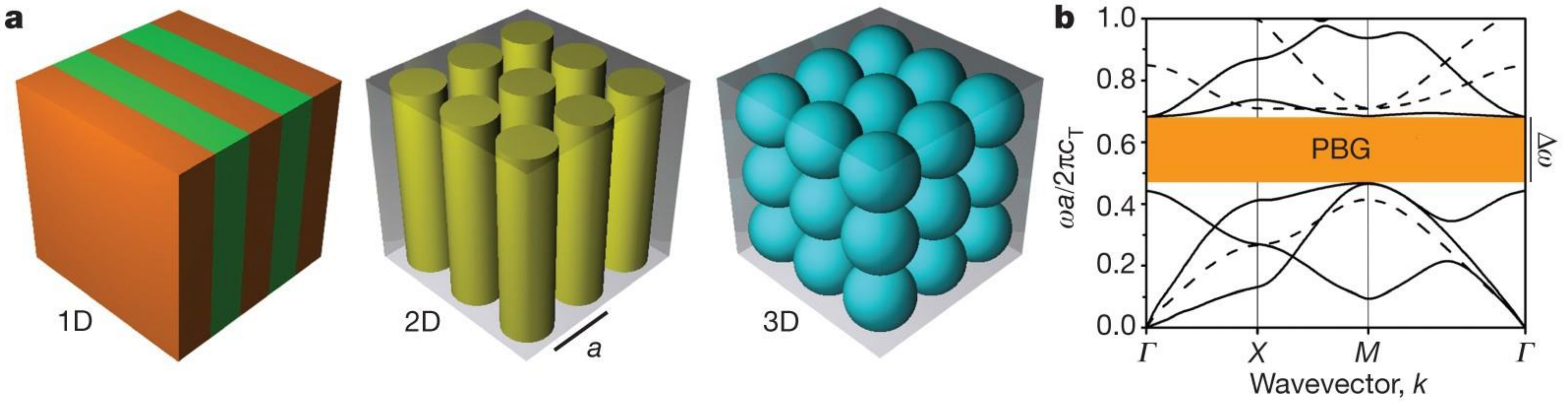
Electrical tuning of elastic wave propagation in nanomechanical lattices at MHz frequencies, *Nature Nanotechnology* **13**, 1016-1020 (2018)

Research Example of Waves in a One-Dimensional Lattice



Electrical tuning of elastic wave propagation in nanomechanical lattices at MHz frequencies, *Nature Nanotechnology* **13**, 1016-1020 (2018)

Waves in a Multi-Dimensional Lattice: Phononic Crystal



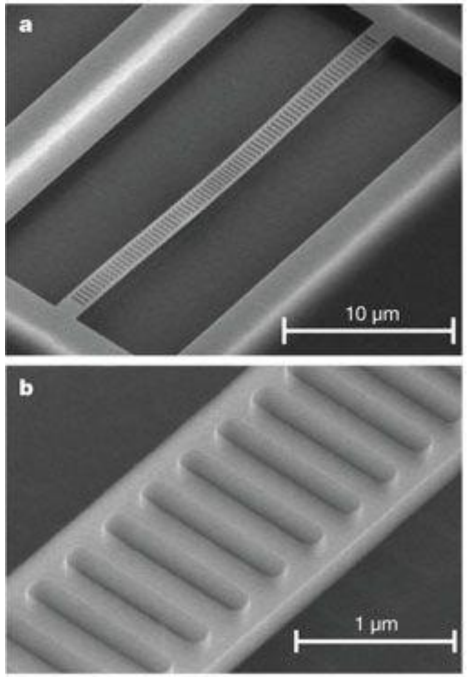
Nature **503** 209–217 (2013)

In general, periodic mechanical systems can be arranged in two- and three-dimensions. Such mechanical systems have received great attention due to the possibility of controlling elastic and acoustic wave propagation in desired ways.

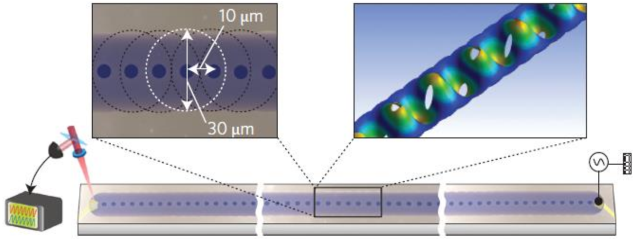
Phononic crystals and metamaterials designate such kinds of mechanical systems nowadays and provide a wide range of opportunities for engineering mechanical waves with many design parameters such as **crystalline symmetries, unit cell architecture, the properties and compositions of constituent materials** and so on.

Waves in a Multi-Dimensional Lattice: Phononic Crystal

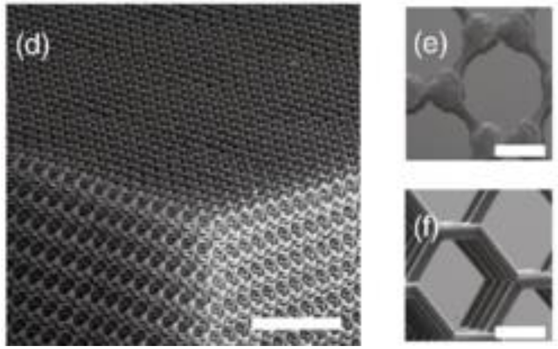
Frequency



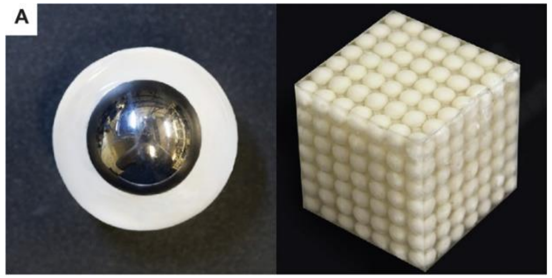
Nature **462**, 78-82 (2009)



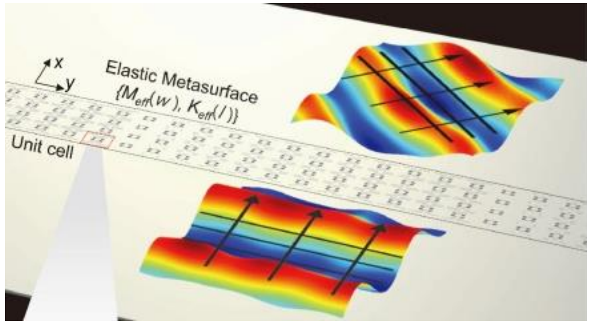
Nat. Nanotech. **9**, 520-524 (2014)



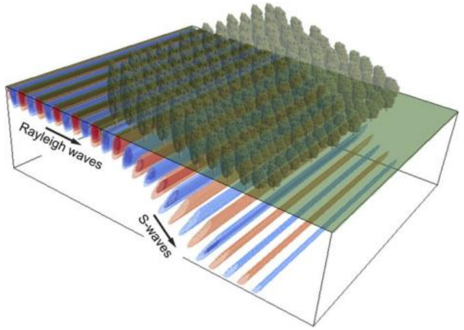
Phys. Rev. App. **6**, 064005 (2016)



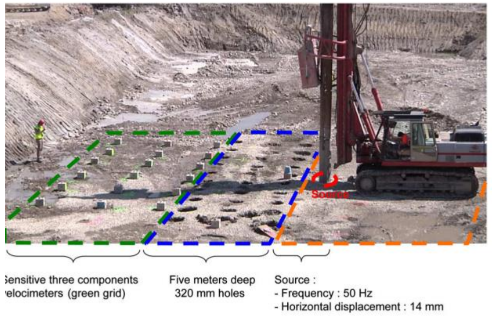
Science **289**, 1734-1736 (2000)



JMPS **112**, 577-593 (2018)



Scientific Reports **6**, 27717 (2016)

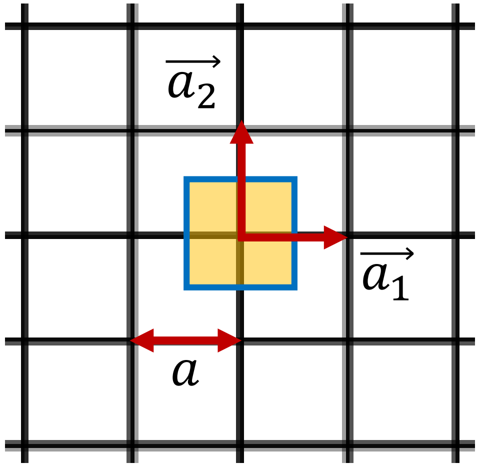


Phys. Rev. Lett **112**, 133901 (2014)

Dimension

Waves in a Multi-Dimensional Lattice: Crystalline Symmetry

square lattice

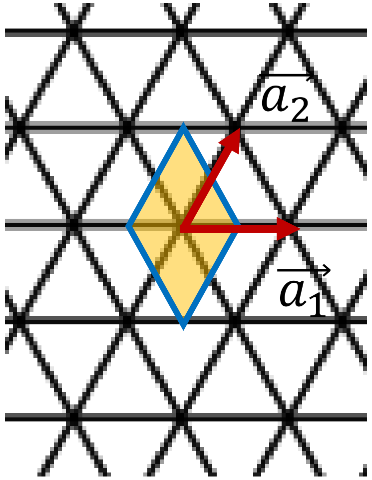


basis vectors:

$$\vec{a_1} = a\hat{\boldsymbol{x}}$$

$$\vec{a_2} = a\hat{\boldsymbol{y}}$$

triangular lattice

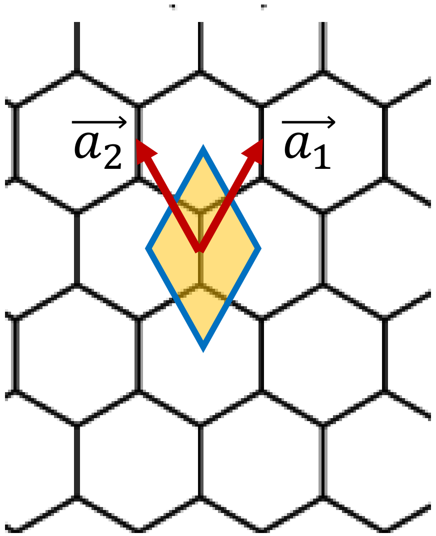


basis vectors:

$$\vec{a_1} = a\hat{\boldsymbol{x}}$$

$$\vec{a_2} = \frac{a}{2}\hat{\boldsymbol{x}} + \frac{\sqrt{3}}{2}\hat{\boldsymbol{y}}$$

honeycomb lattice

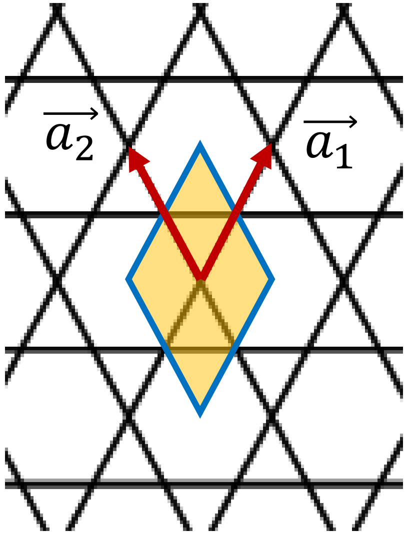


basis vectors:

$$\vec{a_1} = \frac{a}{2}\hat{\boldsymbol{x}} + \frac{\sqrt{3}}{2}\hat{\boldsymbol{y}}$$

$$\vec{a_2} = -\frac{a}{2}\hat{\boldsymbol{x}} + \frac{\sqrt{3}}{2}\hat{\boldsymbol{y}}$$

kagome lattice



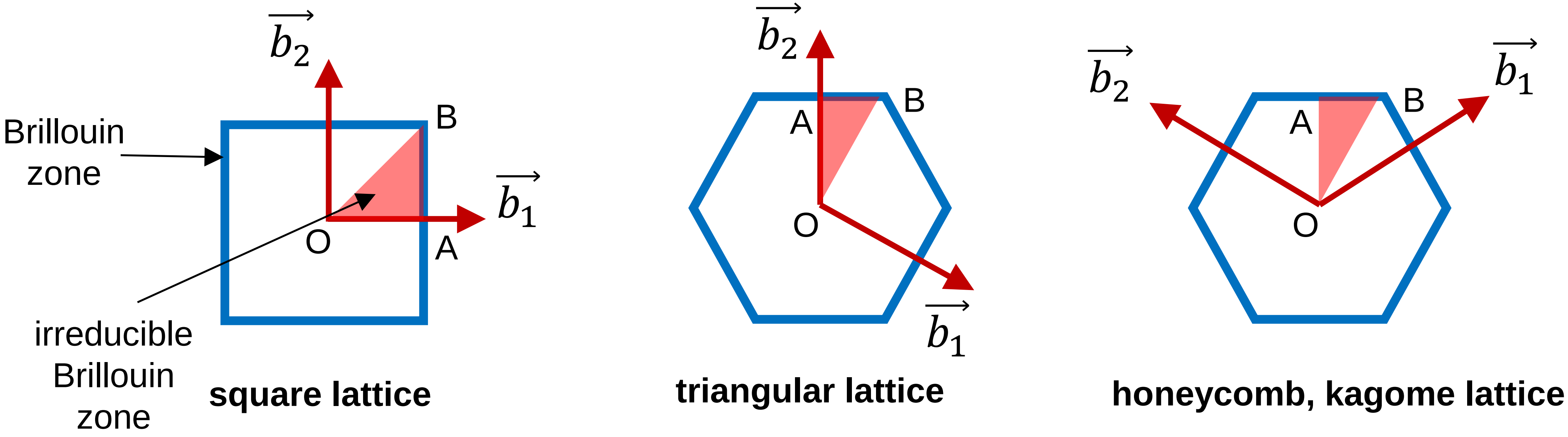
basis vectors:

$$\vec{a_1} = \frac{a}{2}\hat{\boldsymbol{x}} + \frac{\sqrt{3}}{2}\hat{\boldsymbol{y}}$$

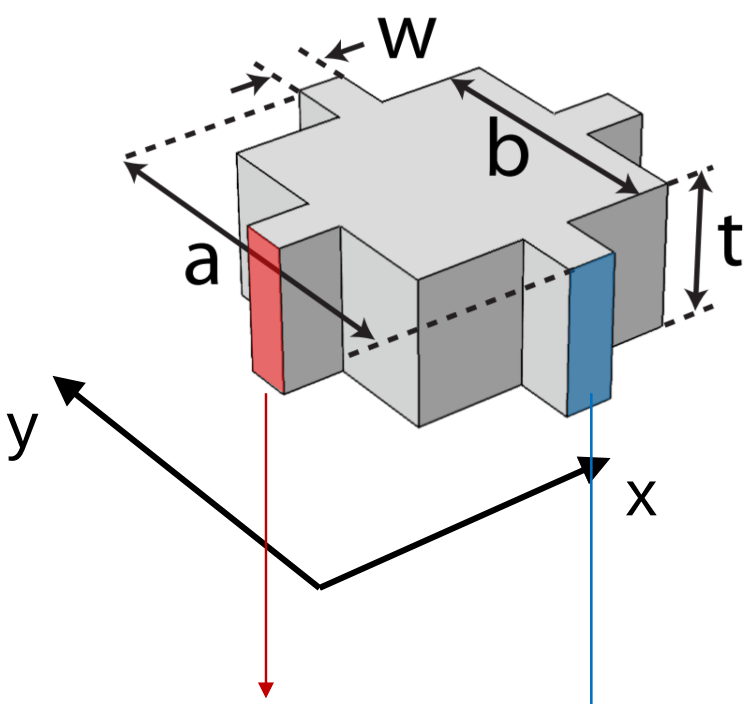
$$\vec{a_2} = -\frac{a}{2}\hat{\boldsymbol{x}} + \frac{\sqrt{3}}{2}\hat{\boldsymbol{y}}$$

Waves in a Multi-Dimensional Lattice: Reciprocal Space

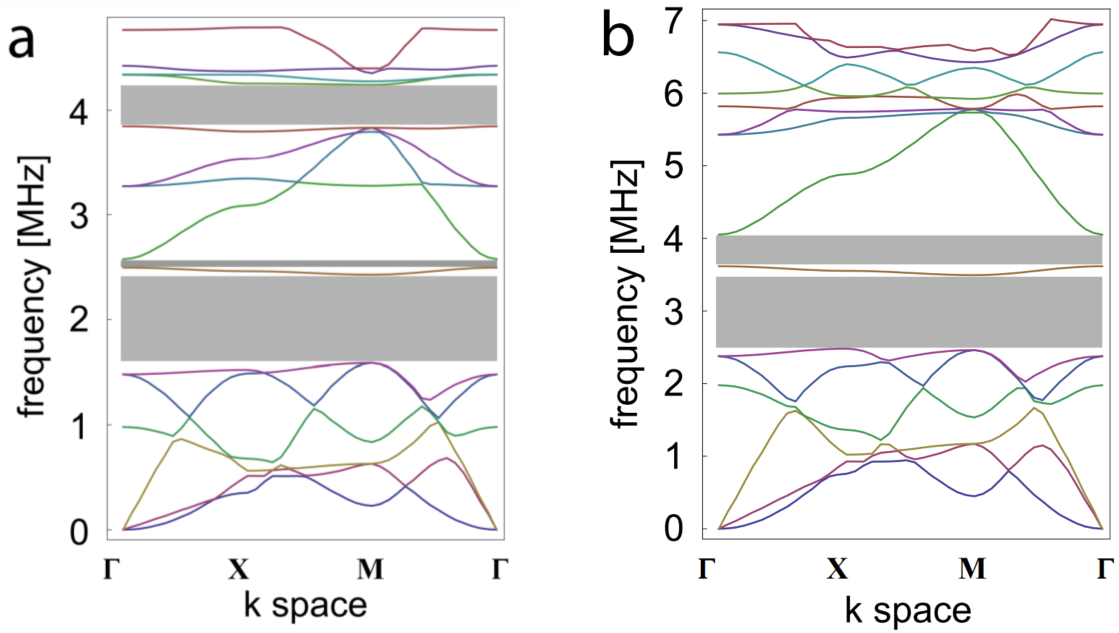
- To study waves in a periodic structure, we analyze its dispersion relation (frequency-wavelength relation) in a reciprocal space or wavevector space.
- When studying mechanical vibrations, we analyze the responses of mechanical systems in frequency domain which is the reciprocal space of time domain.
- However, since a spatial domain can also be two- and three-dimensional unlike time domain (which is one-dimensional), a reciprocal space has the same dimension with that of a space domain.
- Lattices in real space have various crystalline symmetries, so we have to find proper reciprocal space to study the behavior of waves in the lattices.



Research Example 1 of Phononic Crystals



band structure analysis



$$\vec{u}(x+a, y) = \vec{u}(x, y)e^{ik_x a}$$

$$\vec{u}(x, y+a) = \vec{u}(x, y)e^{ik_y a}$$

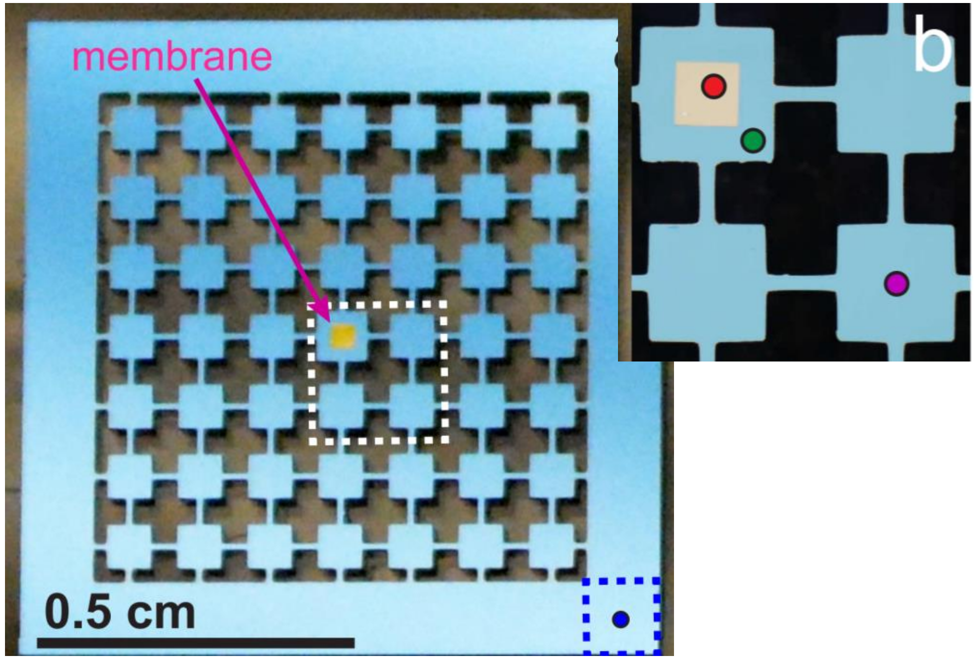
Bloch periodic conditions!

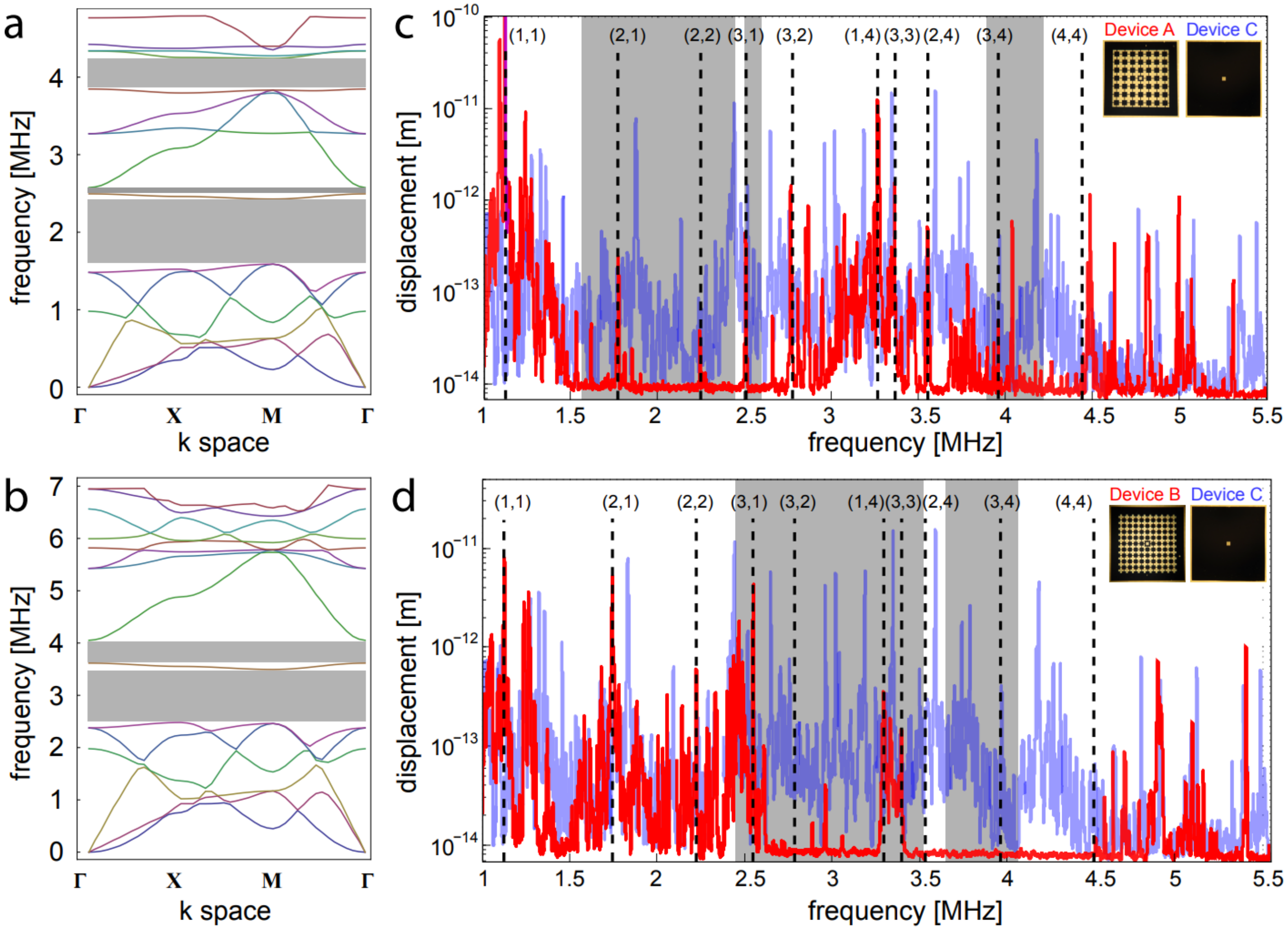
$\vec{u}(x, y)$: displacement field

a = 1100 um	a = 800 um
b = 686 um	b = 542 um
w = 97 um	w = 96 um
t = 300 um	t = 300 um

A phononic bandgap shield for high-Q membrane microresonators, *Applied Ph ysics Letters* **104**, 023510 (2014)

Research Example 1 of Phononic Crystals



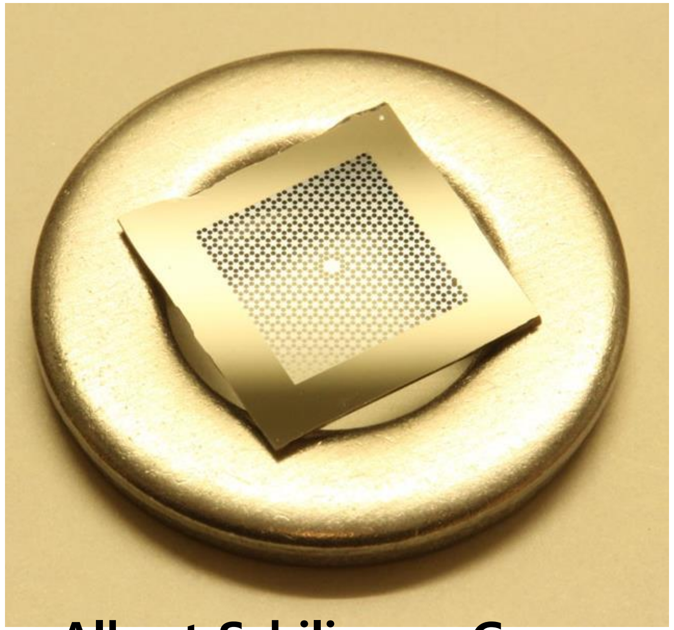


Here, the authors utilized phononic band gaps (of the silicon frame) to realize high-Q nanomechanical vibrations !

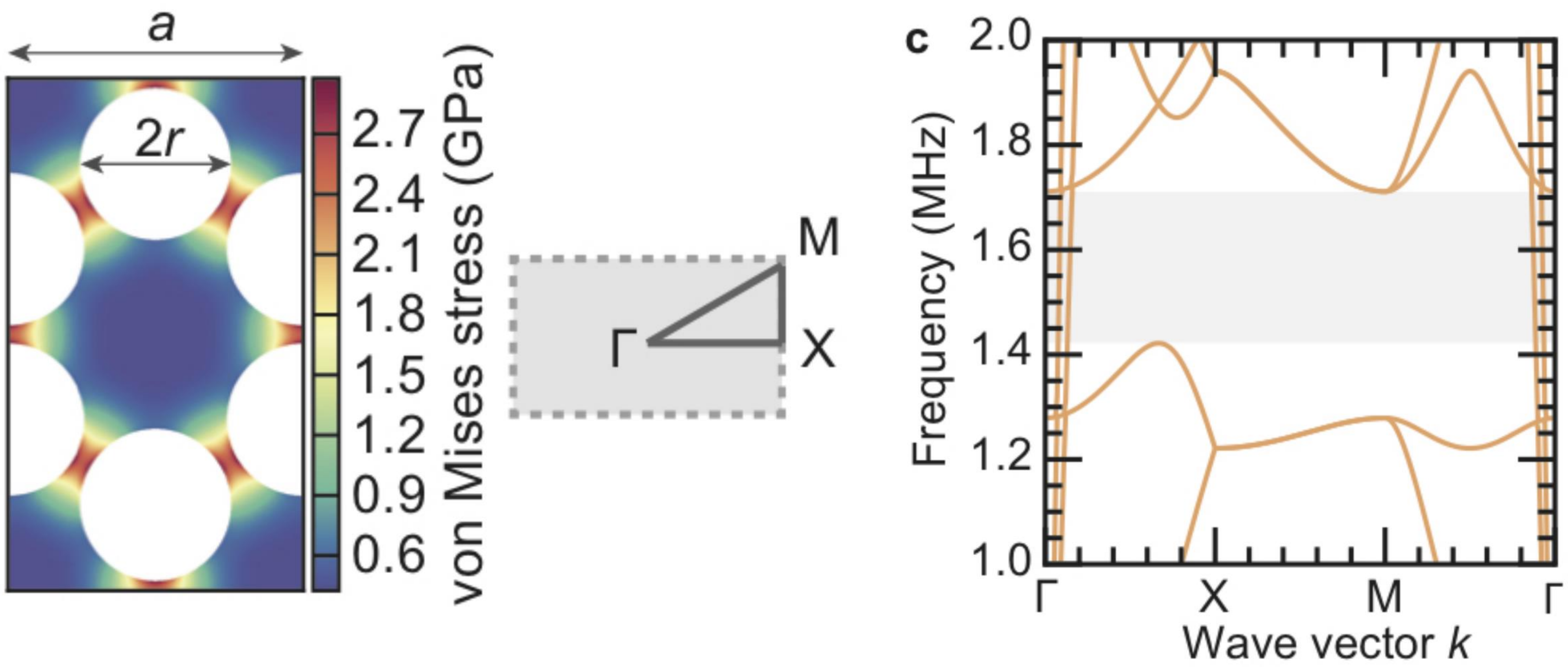
We will see later how this is related to quantum technology

A phononic bandgap shield for high-Q membrane microresonators, *Applied Physics Letters* **104**, 023510 (2014)

Research Example 2 of Phononic Crystals



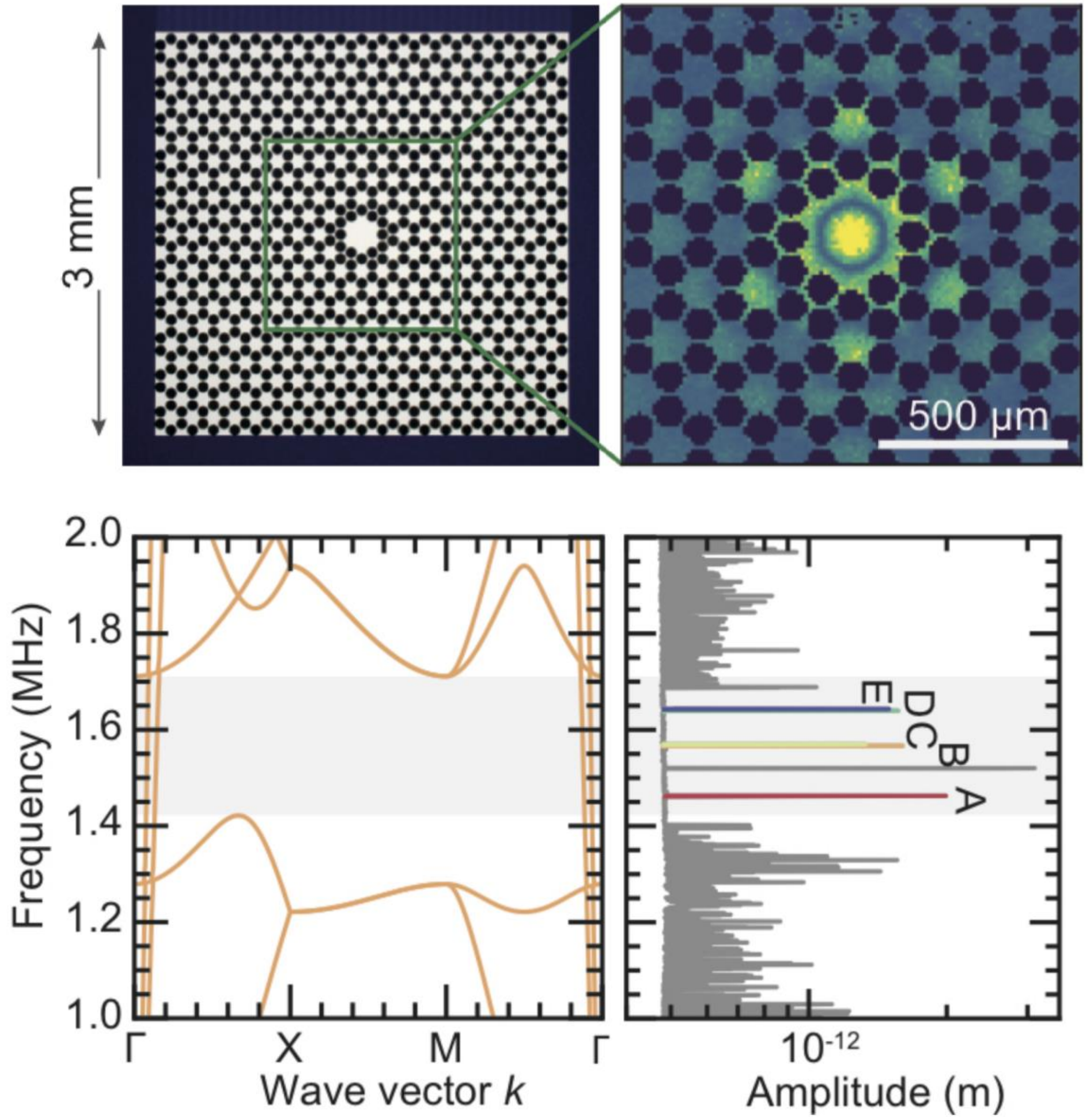
Albert Schiliesser Group
at the Niels Bohr Institute

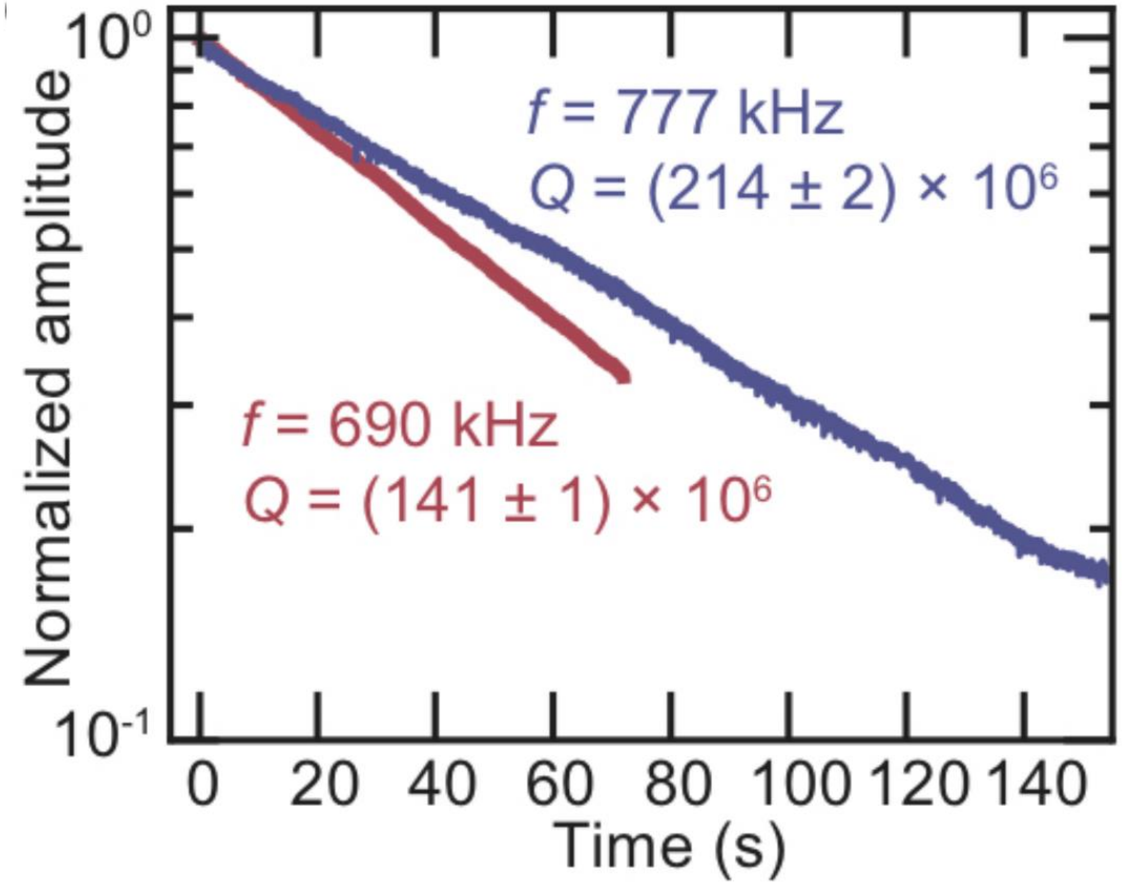


In this research, they realize a phononic crystal in a highly stressed silicon nitride nanomembrane (σ~ 1.27 GPa) and utilized the phononic band gap to achieve ultra high-Q mechanical resonator (Q~ 10^8)

Ultracoherent nanomechanical resonators via soft clamping and dissipation dilution
, *Nature Nanotechnology* **12**, 776-783 (2017)

Research Example 2 of Phononic Crystals





Ultracoherent nanomechanical resonators via soft clamping and dissipation dilution , *Nature Nanotechnology* **12**, 776-783 (2017)

Simulation of Phononic Crystals using COMSOL

Step 1. Design the geometry and define your primitive unit cell of the lattice and define the material pr operties

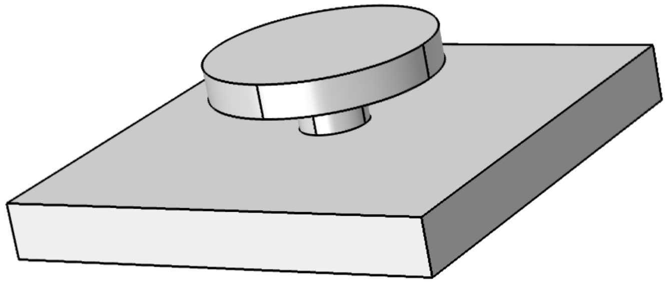
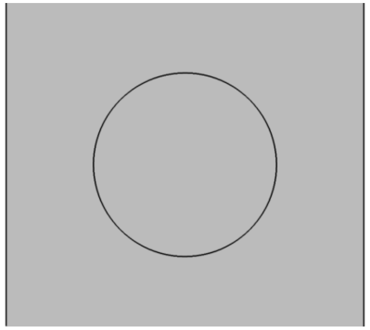
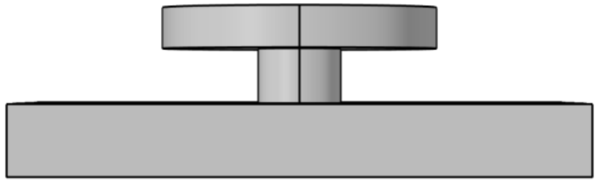


Plate with a resonator

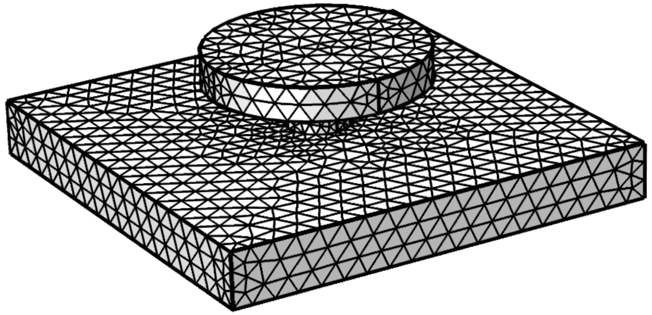


Top view



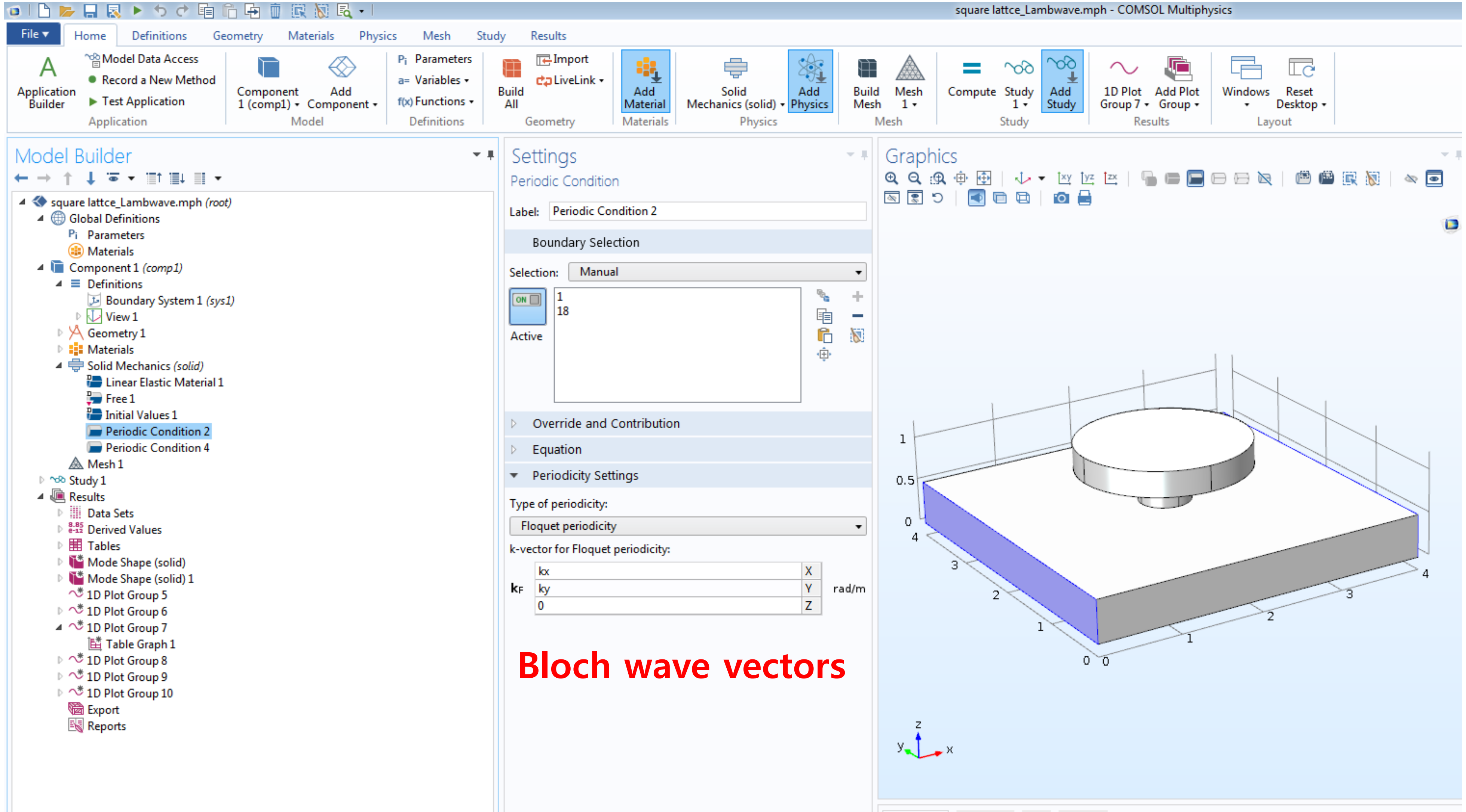
Side view

Step 2. Mesh generation: discretization of your system, constructing mass and stiffness matrices



Simulation of Phononic Crystals using COMSOL

Step 3. Apply the Bloch periodic conditions(select Floquet periodicity in COMSOL)



Simulation of Phononic Crystals using COMSOL

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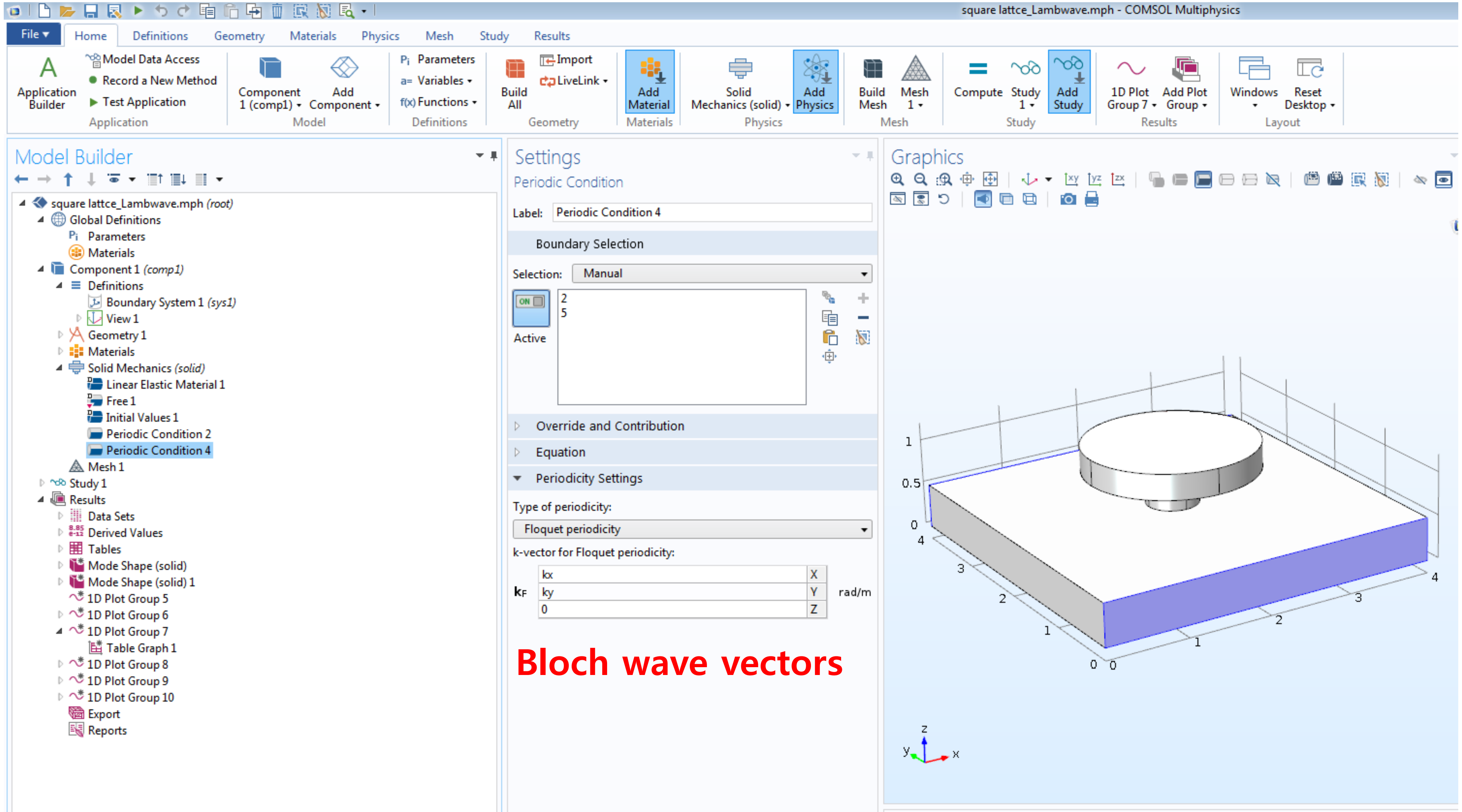
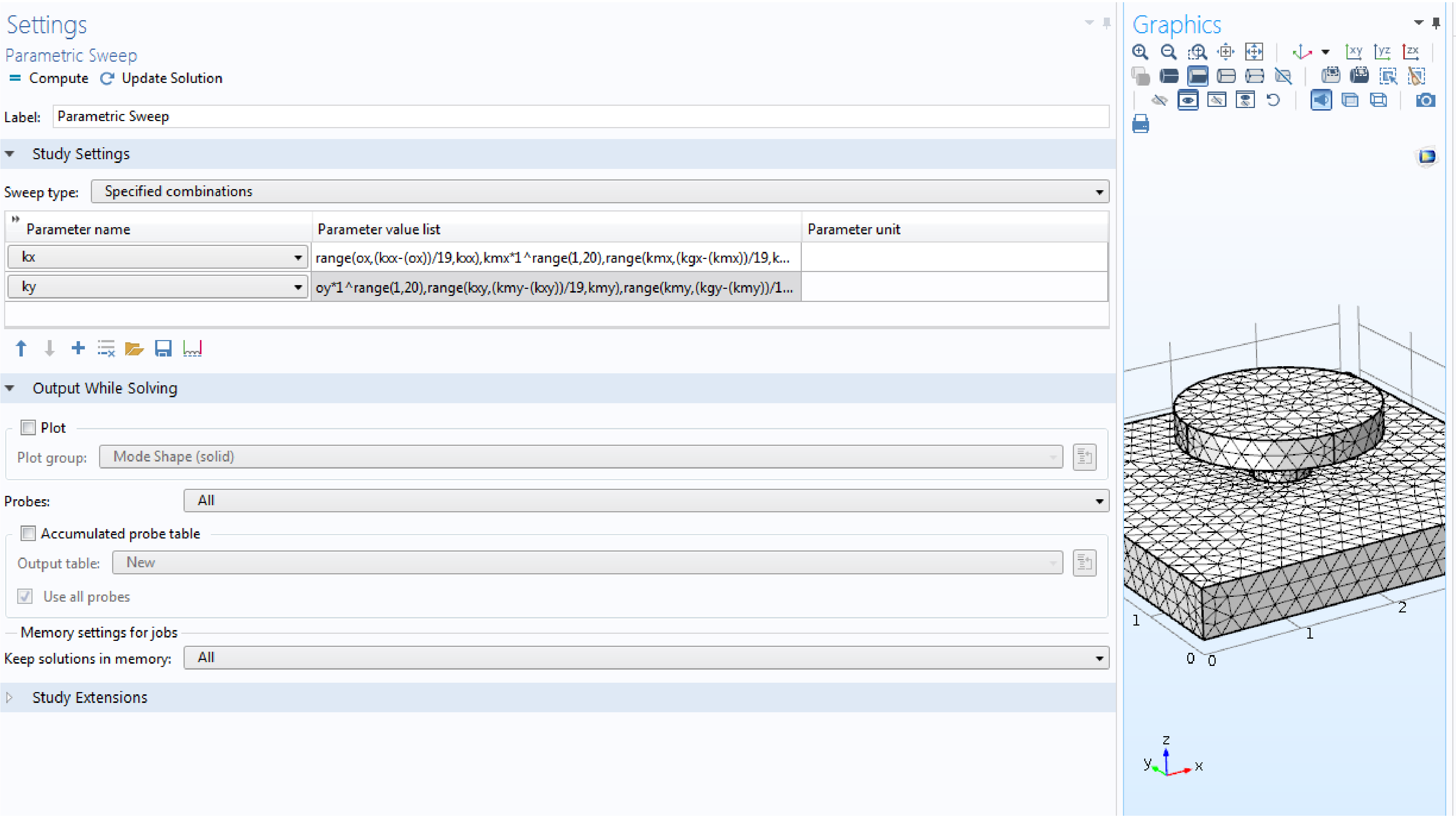


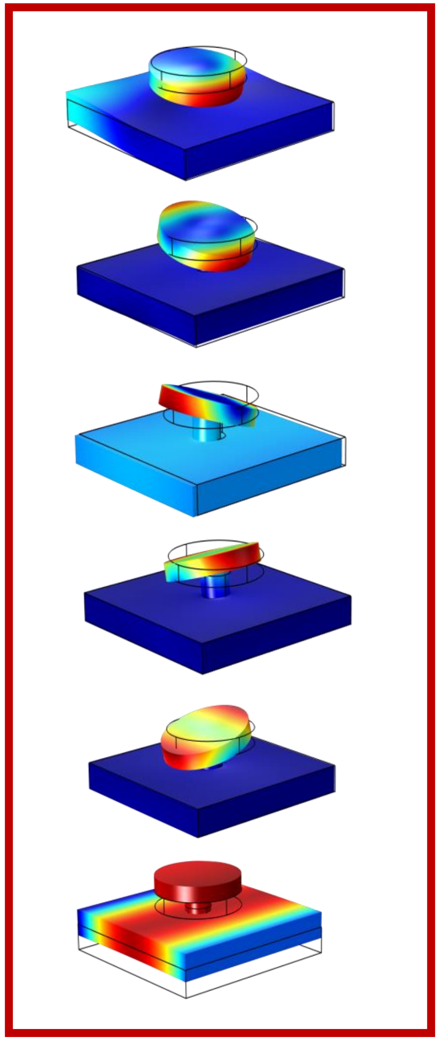
Simulation of Phononic Crystals using COMSOL

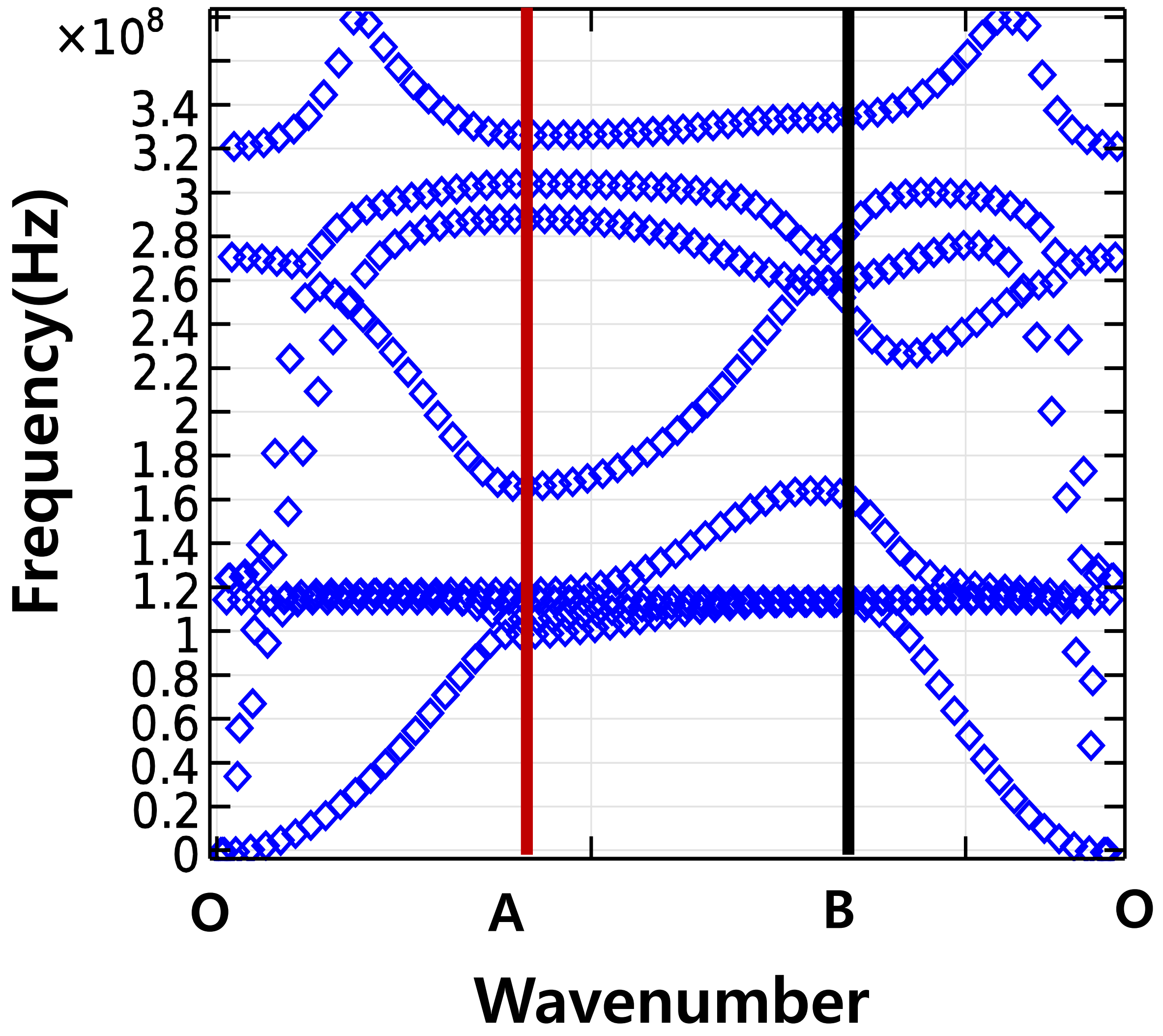
Step 4. Solve the eigenvalue equations for different wave vectors(at the boundary of irreducible BZ)

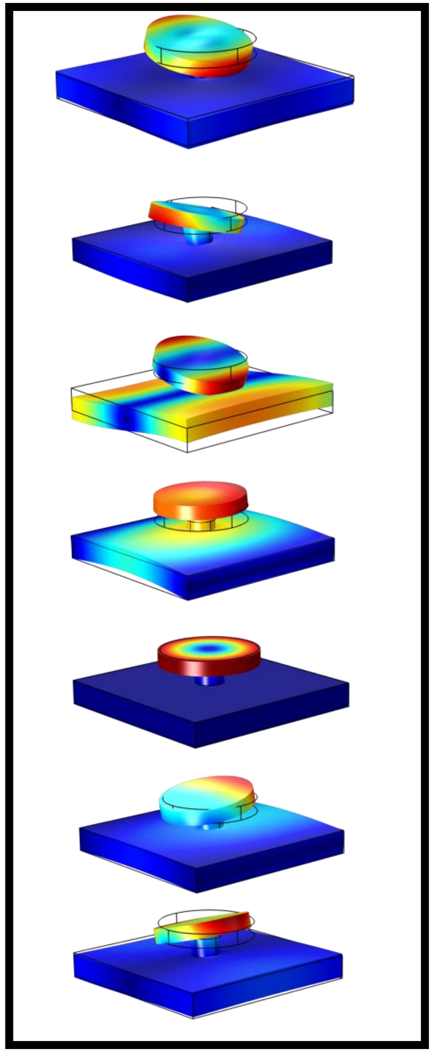


Simulation of Phononic Crystals using COMSOL

Step 5. Plot your dispersion curves and analyze mode dynamics!







Let's Do Some Quantum from Now On

Quantum Mechanics of A Single-Mode Mechanical Resonator

A time-independent Schroedinger equation reads

$$\hat{H}\psi(x) = E\psi(x)$$

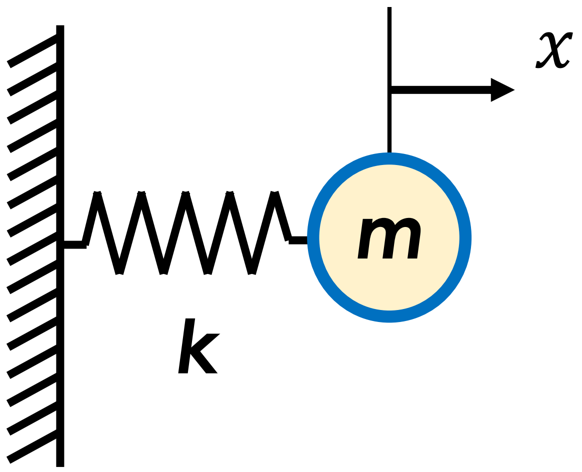
$\hat{H}$: **Hamiltonian operator**

ψ: **wave function(eigenfunction)**

E: **energy(eigenvalue)**

$$\int |\psi(x)|^2 \, dx = 1$$

$|\psi(x)|^2$: **the probability of finding the particle at point x**



Recall the Hamiltonian for the mechanical oscillator

$$\hat{H} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$$

$p = mv$: **momentum operator**

x: **position operator**

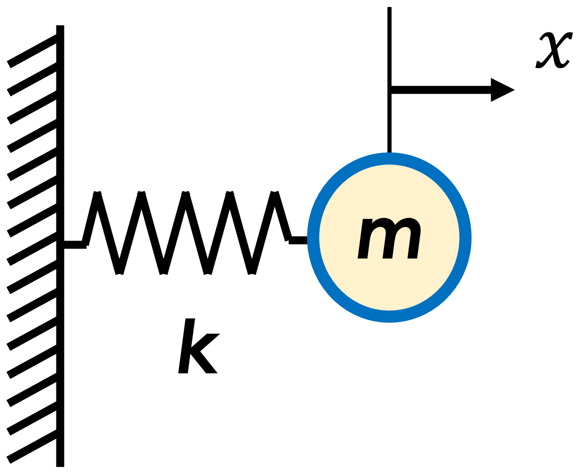
$k = m\omega^2$: **spring constant**

ω: **(angular)resonant frequency**

Quantum Mechanics of A Single-Mode Mechanical Resonator

$$\hat{H}\psi(x) = E\psi(x)$$

$$\left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right)\psi(x) = E\psi(x)$$



If we define new operators

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x})$$

creation operator

(생성)

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega\hat{x})$$

annihilation operator

(소멸)

$$\hat{H}\psi(x) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\psi(x) = E\psi(x)$$

Phonon:

Phonon is a quasiparticle(준입자) which represents a quantized energy unit of mechanical vibrations and waves. It is like photon for electromagnetic waves.

Ex) we have many phonons in a mechanical resonator => the energy of a mechanical resonator is high.

Quantum Mechanics of A Single-Mode Mechanical Resonator

$$\hat{H}\psi(x) = E\psi(x)$$

$\hat{a}^\dagger$ **creation operator**

$\hat{a}$ **annihilation operator**

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

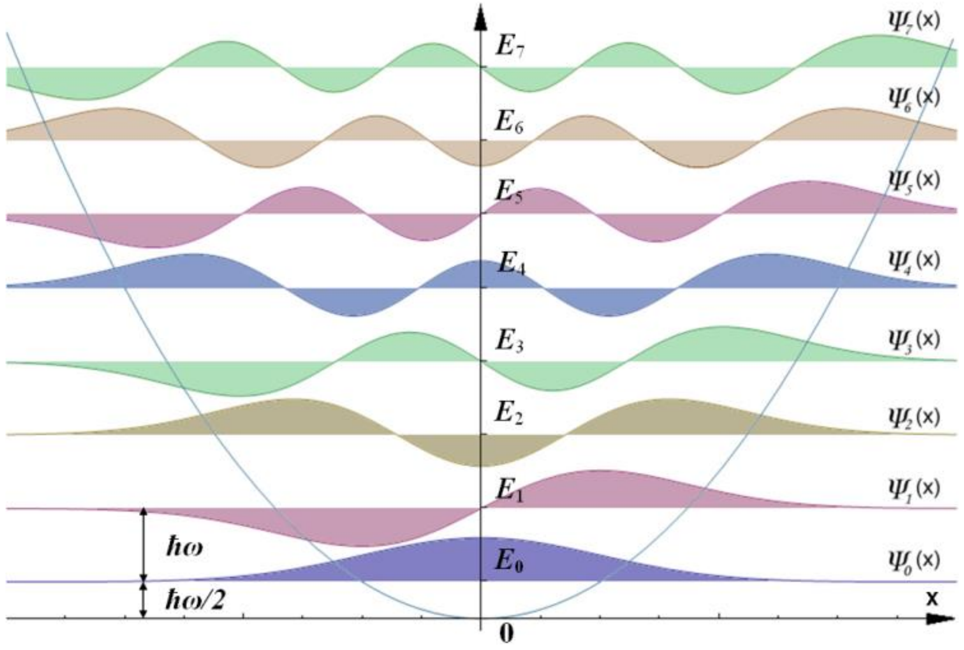
Hamiltonian for a single-mode mechanical resonator

$$E = \hbar\omega\left(n + \frac{1}{2}\right)$$

Energy of a single-mode mechanical resonator

$$x_{zpf} = \sqrt{\frac{\hbar}{2m\omega}}$$

zero-point fluctuation



Thermodynamic Aspects of A Single-Mode Mechanical Resonator

- Bose-Einstein Distributions: A distribution that shows the average number of particles that occupy a quantum state. Also called the occupancy of the quantum state.

$$n = \frac{1}{e^{\frac{hf}{k_B T}} - 1}$$

h: **Planck's constant**

f: **resonant or mode frequency**

k_B: **Boltzmann constant**

T: **Temperature**

- The exponent $\frac{hf}{k_B T}$ is the most important indicator that informs us of whether a system behaves quantum mechanically or not.

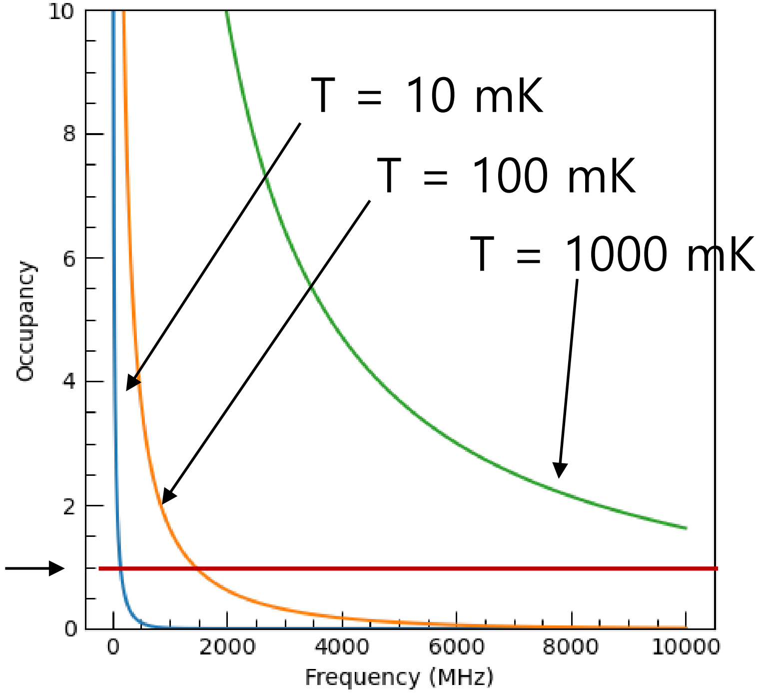
- If $hf \ll k_B T, \quad n \gg 1$ ⟶ Thermal energy(noise) excites the system. A system behaves classically.
- If $hf \gg k_B T, \quad n \ll 1$ ⟶ No boson exists in the system. The system is in the quantum ground state or is quantum vacuum. A system behaves quantum mechanically.

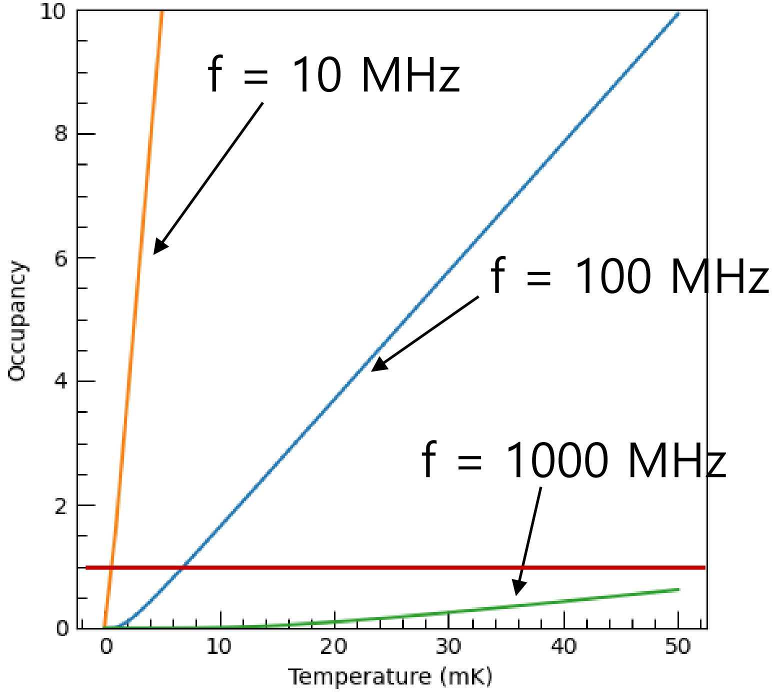
Thermodynamic Aspects of A Single-Mode Mechanical Resonator

- Let's look at the Bose-Einstein distribution for different temperatures and frequencies.

$$n = \frac{1}{e^{\frac{hf}{k_B T}} - 1}$$

single phonon →





Note:

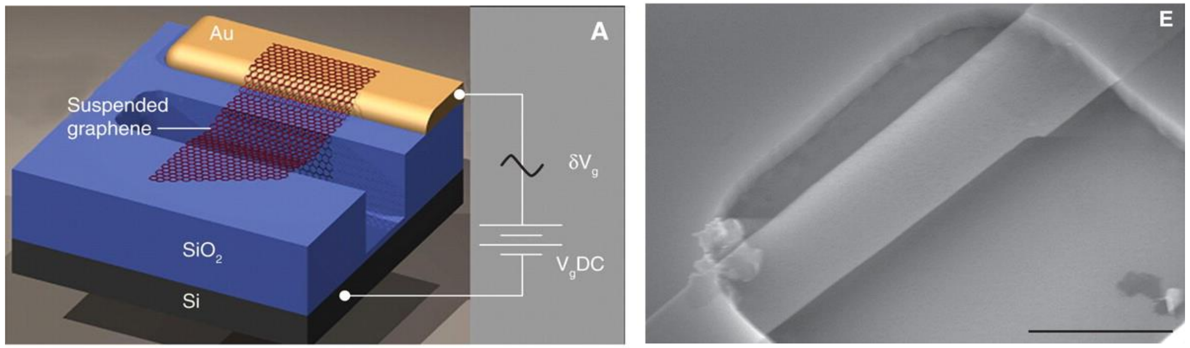
Superconducting quantum devices operate at mK temperatures!

- The occupancy decreases as the temperature decreases at the same resonant frequency.
- The occupancy decreases as the frequency increases at the same temperature.

Thermodynamic Aspects of A Single-Mode Mechanical Resonator

Examples) Calculate the phonon occupancy of two different nanomechanical systems oscillating at different resonant frequencies at 20 mK.

Graphene nanomechanical resonator $f \sim 20\ MHz$

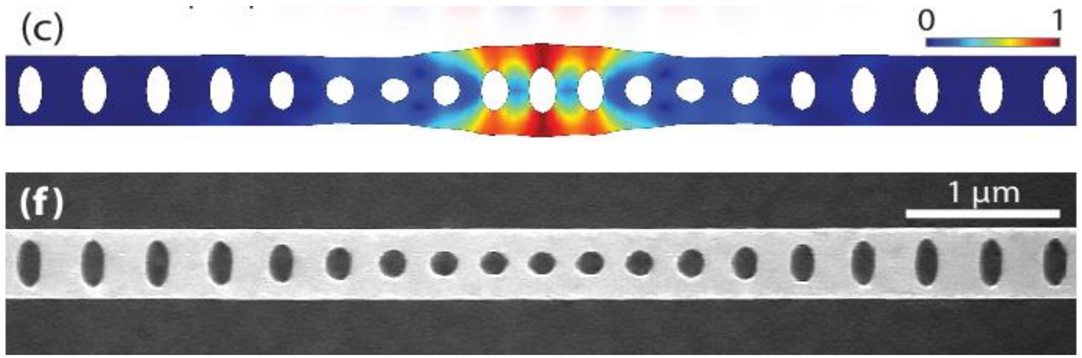


J. S. Bunch *et al. Science* **315,** 490-493 (2007)

$$n = \frac{1}{e^{\frac{hf}{k_B T}} - 1} = \frac{1}{e^{\frac{(6.626\times10^{-34} J\cdot s)(20\ MHz)}{(1.38\times10^{-23} J/K)(20\ mK)}} - 1} \approx 20$$

There are **20 phonons** thermally created!
=> The thermally excited mechanical resonator has its energy corresponding to 20 phonons!

Silicon Phonon Cavity $f \sim 3\ GHz$

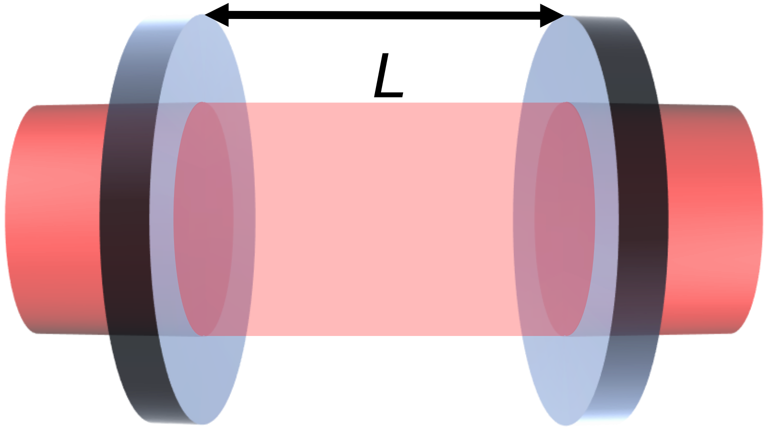


J. Chan *et al. Appl. Phys. Lett.* **101,** 081115 (2012)

$$n = \frac{1}{e^{\frac{hf}{k_B T}} - 1} = \frac{1}{e^{\frac{(6.626\times10^{-34} J\cdot s)(3\ GHz)}{(1.38\times10^{-23} J/K)(20\ mK)}} - 1} \approx 0.0007$$

There are almost **no phonon** in the system!
=> The mechanical resonator are now in its quantum ground state!

Quantum Mechaics of A Single-Mode Electromagnetic Resonator



an optical cavity(Fabry-Perot type)
made of two highly reflective mirrors

Remark) Optical cavities are essential components for cavity quantum electrodynamics(cavity QED) and thus quantum information science!

We start from the well-known Maxwell's equations. Note that there is no charge or current source that provides additional electric or magnetic fields.

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

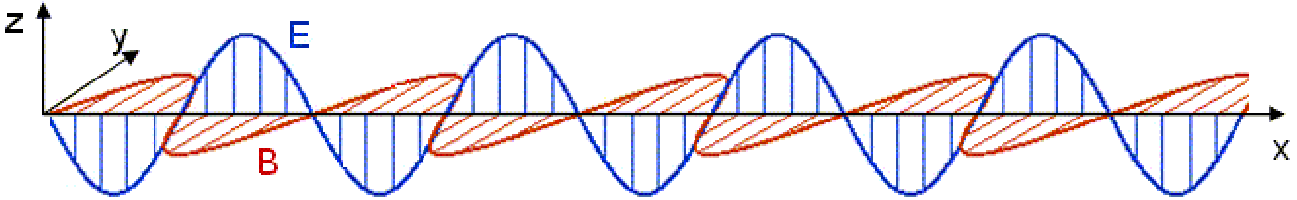
$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$\vec{E}$: **electric field**

$\vec{B}$: **magnetic field**

μ_0: **vacuum permeability**

ε_0: **vacuum permittivity**

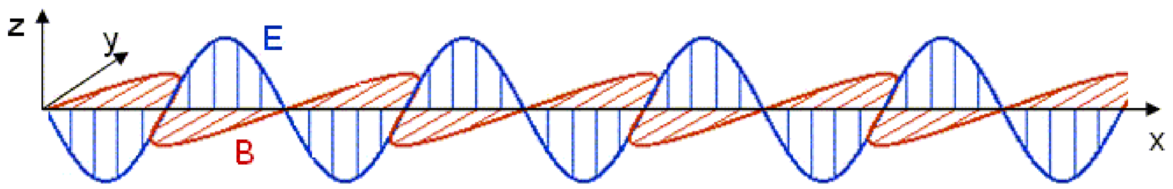


Quantum Mechaics of A Single-Mode Electromagnetic Resonator

From the four Maxwell's equations, we derive the wave equation for electromagnetic waves in vacuum. This reads

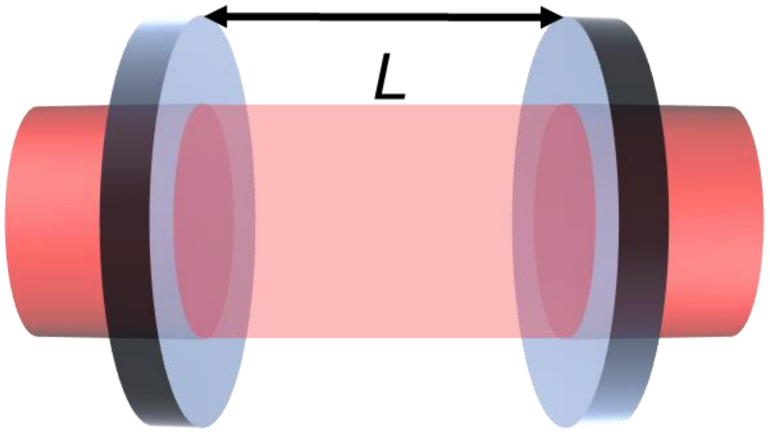
$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{with} \quad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}:$$ **the speed of light**

Assume that the electric field is linearly polarized in z-direction as in the figure and the propagation direction is parallel to x-direction.



The total energy of the electromagnetic field contained in the cavity is given by

$$U = \frac{1}{2}\int_V \left(\varepsilon_0 E_z^2 + \frac{B_y^2}{\mu_0}\right) dV$$



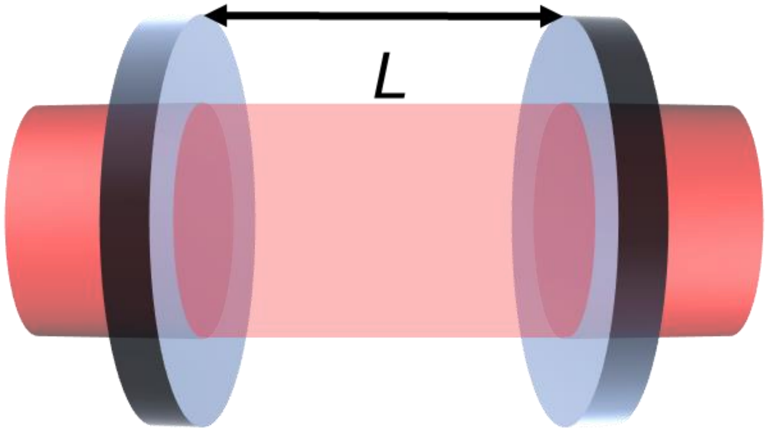
Quantum Mechaics of A Single-Mode Electromagnetic Resonator

If we let $E_z = q(t)\sin kx$, we obtain $B_y = -\frac{\mu_0\varepsilon_0}{k}\dot{q}(t)\cos kx$

Note that we use $\nabla \times \vec{B} = \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$.

If we insert the electric field and the magnetic field into the energy equation, we obtian $U = \frac{1}{2}\int_V \left(\varepsilon_0 E_z^2 + \frac{B_y^2}{\mu_0}\right) dV$

$$U = \frac{\varepsilon_0 V}{2}\left[\frac{\dot{q}^2(t)}{c^2k^2} + q^2(t)\right] = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$



Here, we define a momentum $p = mq$ with $m = \frac{\varepsilon_0 V}{\omega^2}$.

The form of the equation remind us of the energy of a single-mode mechanical resonator!

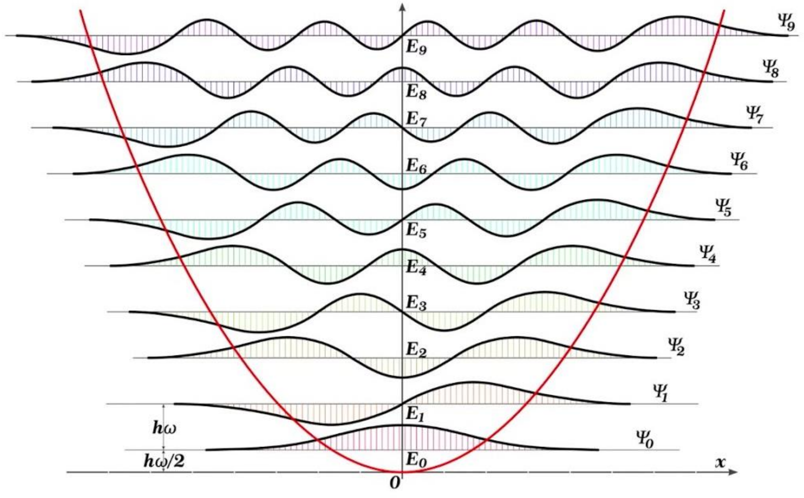
Quantum Mechaics of A Single-Mode Electromagnetic Resonator

Therefore, electromagnetic waves behaves like mechanical resonators in quantum mechanics and their quantization is called **photons**. So all the quantum mechanical definitions of the mechanical resonators are also valid for photons.

$$\hat{H} = \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right)$$

$\hat{a}^{\dagger}$ **creation operator**

$\hat{a}$ **annihilation operator**



$$n = \frac{1}{e^{\frac{hf}{k_B T}} - 1}$$

Thermal Occupancy

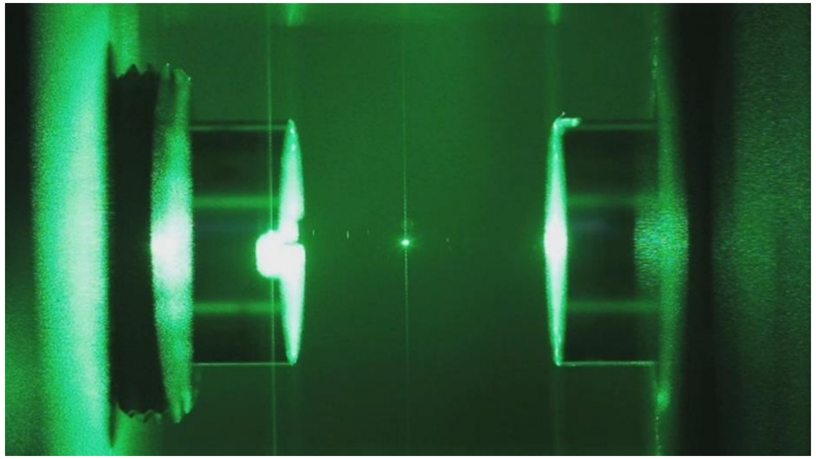
$$x_{zpf} = \sqrt{\frac{\hbar\omega}{2V\varepsilon_0}}$$

zero-point fluctuation

$$E = \hbar\omega\left(n + \frac{1}{2}\right)$$

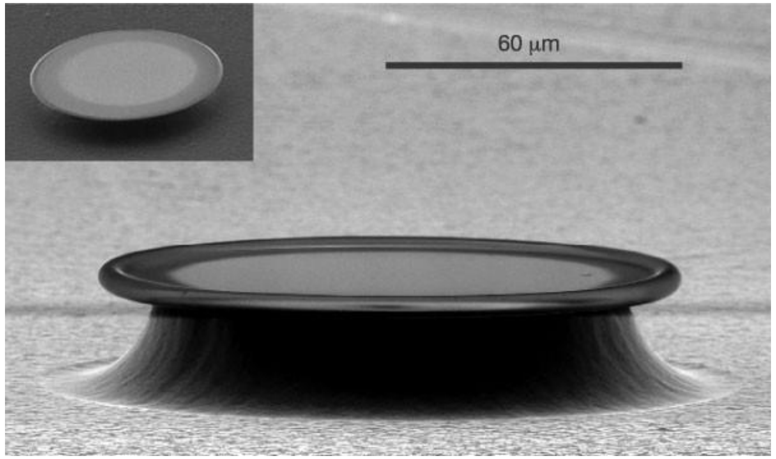
Quantum Mechaics of A Single-Mode Electromagnetic Resonator

There are many types of optical cavities such as free-space optical cavities, whispering gallery mode optical resonators, and photonic crystal cavities.



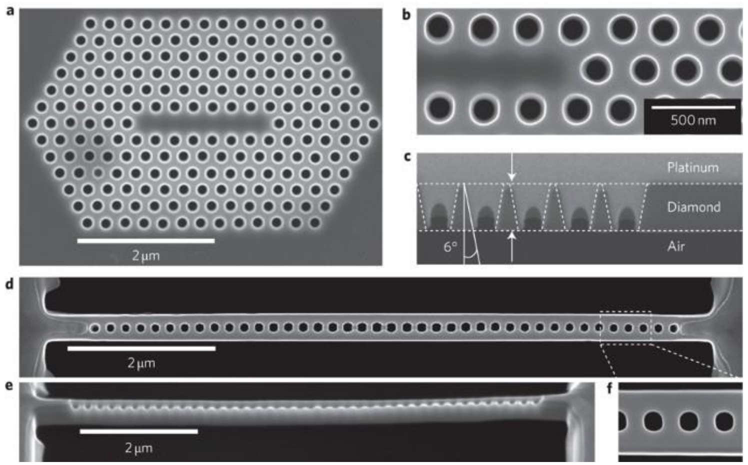
Wikipedia

A free-space optical cavity. This can be used to trap particles and neutral atoms.



D. K. Armani *et al. Nature* **421,** 925-928 (2003)

Whispering gallery mode optical resonator. This can be used to generate optical frequency combs.

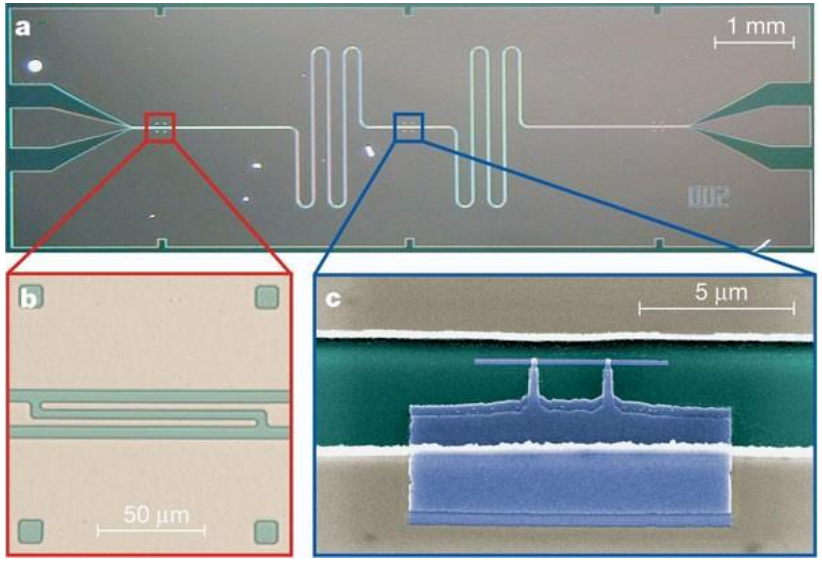


J. Riedrich-Moller *et al. Nat. Nanotechnol.* **7,** 69-74 (2012)

Photonic crystal cavity resonator. This can be used to study cavity QED with NV centers and cavity optomechanics.

Quantum Mechaics of A Single-Mode Electromagnetic Resonator

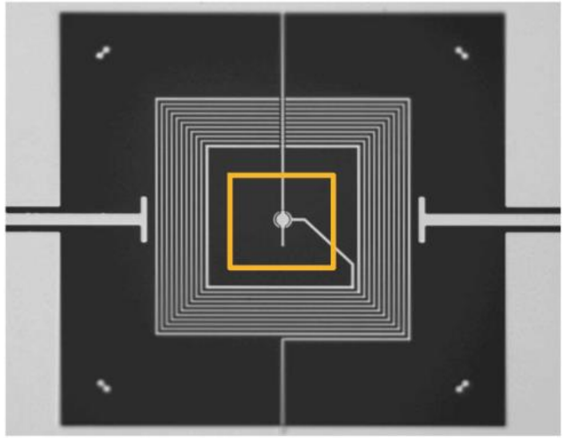
Obviously, microwave resonators follow all the definitions of quantum harmonic oscillators we discussed so far. There are many types of microwave resonators we can realize such as coplanar waveguide cavity, 3D microwave cavity, and LC resonators.

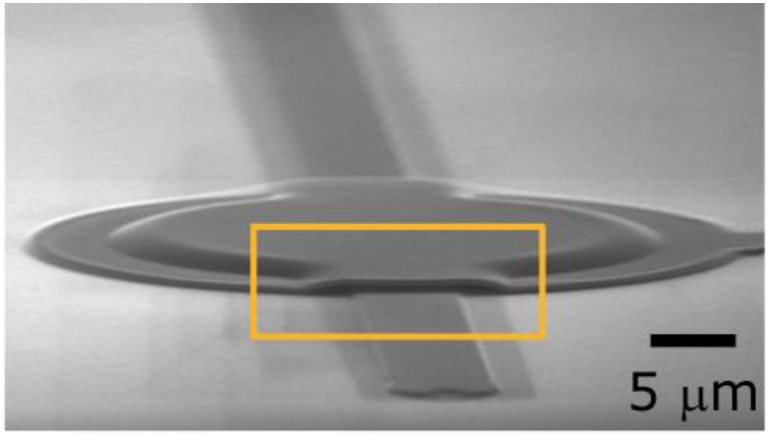


A. Wallraff *et al. Nature* **431,** 162-167 (2004)

coplanar waveguide microwave resonator for qubit measurement.

LC resonator for superconducting nanoelectromechanics



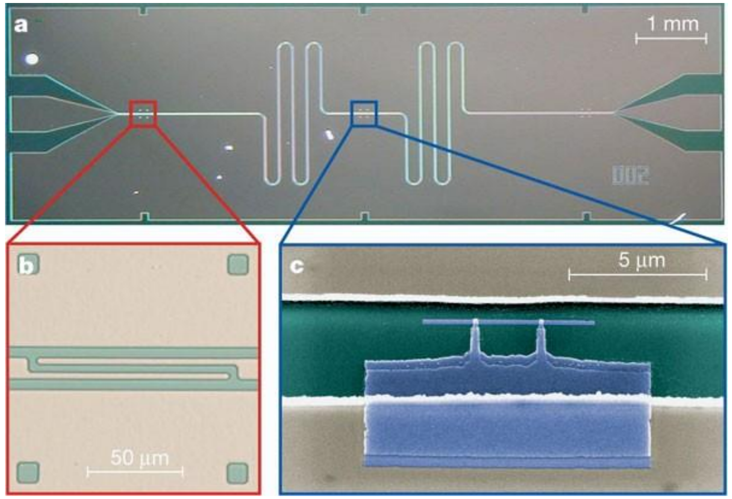


J Cha*, et al. Nano Letters* **21**, 1800-1806 (2021)

Thermodynamics of Electromagnetic Waves

Examples) Calculate the occupancy of two different electromagnetic resonators with different resonant frequencies at 300 K(room temperature).

Superconducting Microwave Resonators $f \sim 7\ GHz$

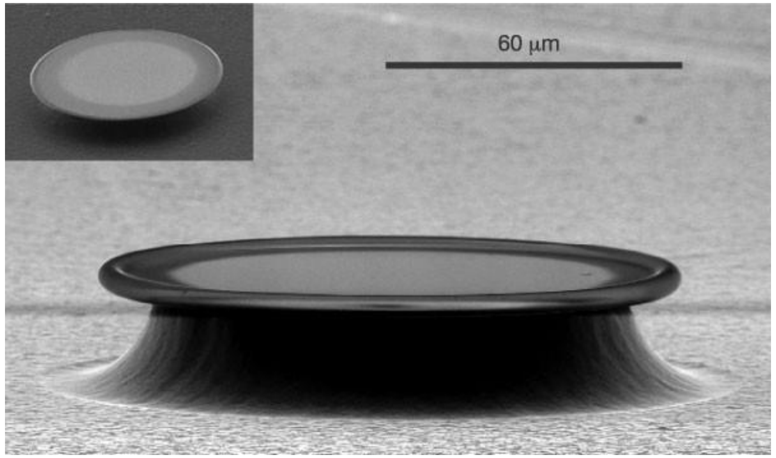


A. Wallraff *et al. Nature* **431,** 162-167 (2004)

$$n = \frac{1}{e^{\frac{hf}{k_B T}} - 1} = \frac{1}{e^{\frac{(6.626\times10^{-34} J\cdot s)(7\ GHz)}{(1.38\times10^{-23} J/K)(300\ K)}} - 1} \approx 892$$

There are **892 photons** thermally created at the room temperature! That's why we have to operate superconducting quantum devices at mK temperatures!

Optical resonator $f \sim 193\ GHz$



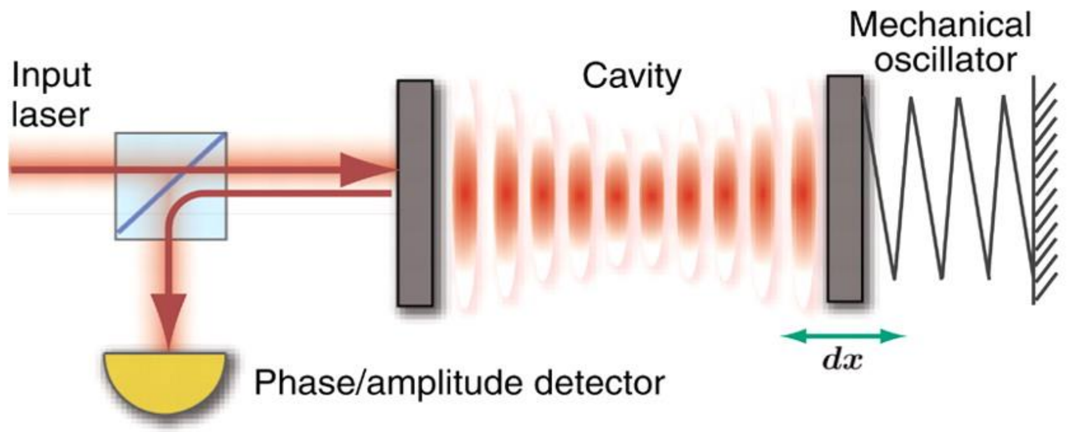
D. K. Armani *et al. Nature* **421,** 925-928 (2003)

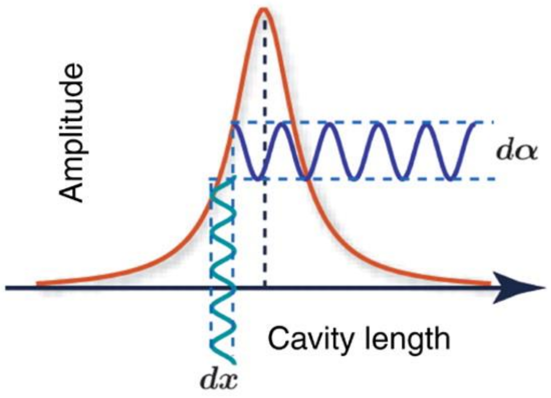
$$n = \frac{1}{e^{\frac{hf}{k_B T}} - 1} = \frac{1}{e^{\frac{(6.626\times10^{-34} J\cdot s)(193\ THz)}{(1.38\times10^{-23} J/K)(300\ K)}} - 1} \approx 4(10^{-14})$$

no photons in the system! The system is in its quantum ground state. That's why photon-based quantum experiments can be realized at the room temperature!

Cavity Optomechanics: Bring Mechanical Vibrations to Quantum Regime

Optical and Mechanical Resonators





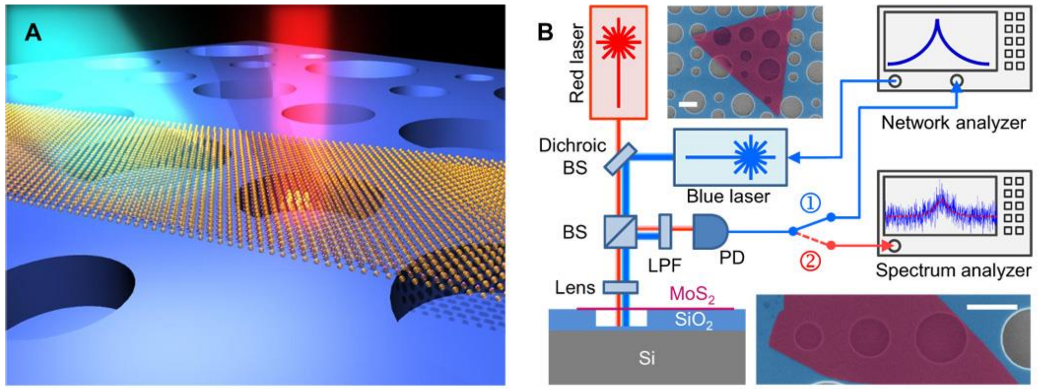
displacement of the mechanical resonator (***dx***)

↓

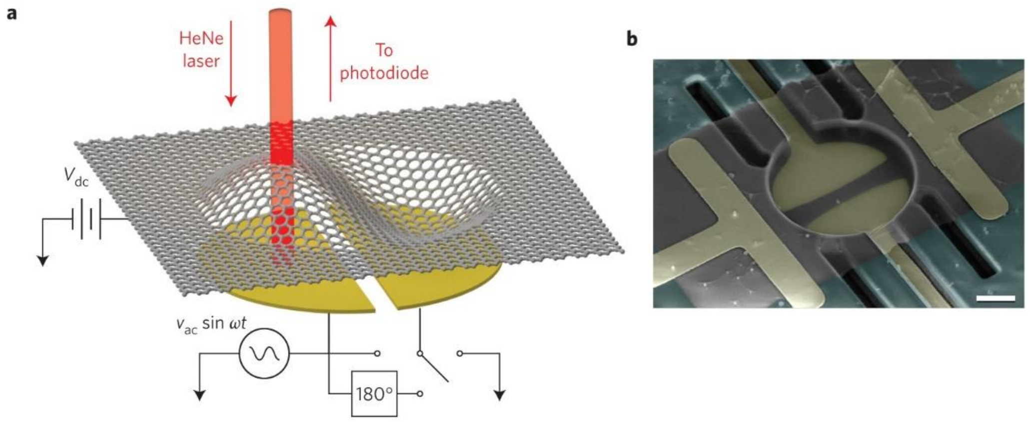
intensity change of the reflected light (***da***)

T. J. Kippenberg, Cavity Optomechanics: Back-Action at the Mesoscale, ***Science*** 321, 1172 - 1176 (2008)

Examples: 2D Nanomechanical Resonators



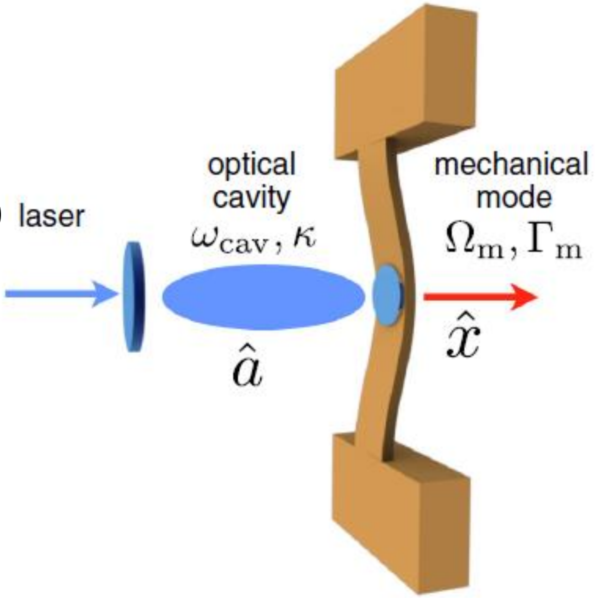
Jaesung Lee *et al.* ***Science Advances*** eaao6653 (2018)

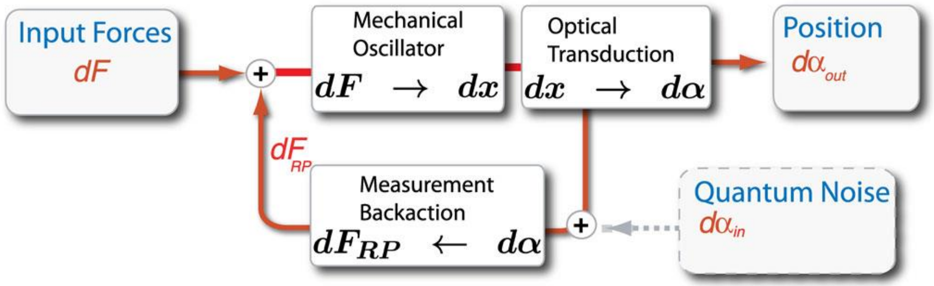


R. De Alba *et al.* ***Nature Nanotechnology*** **11,** 741-746 (2016)

Cavity Optomechanics: Bring Mechanical Vibrations to Quantum Regime

(coupling)





Review of Modern Physics 86, 1391 - 1452 (2014)

- Cavity enhanced photons exert forces (via the radiation-pressure) to the mechanical resonator
- The consequent motion of the mechanical resonator perturbs the optical cavity
- Due to the cavity perturbation, the forces applied by the photons change.

$$\hat{H} = \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\Omega_m \hat{b}^\dagger \hat{b} - \hbar g_0 \hat{a}^\dagger \hat{a}(\hat{b}^\dagger + \hat{b})$$

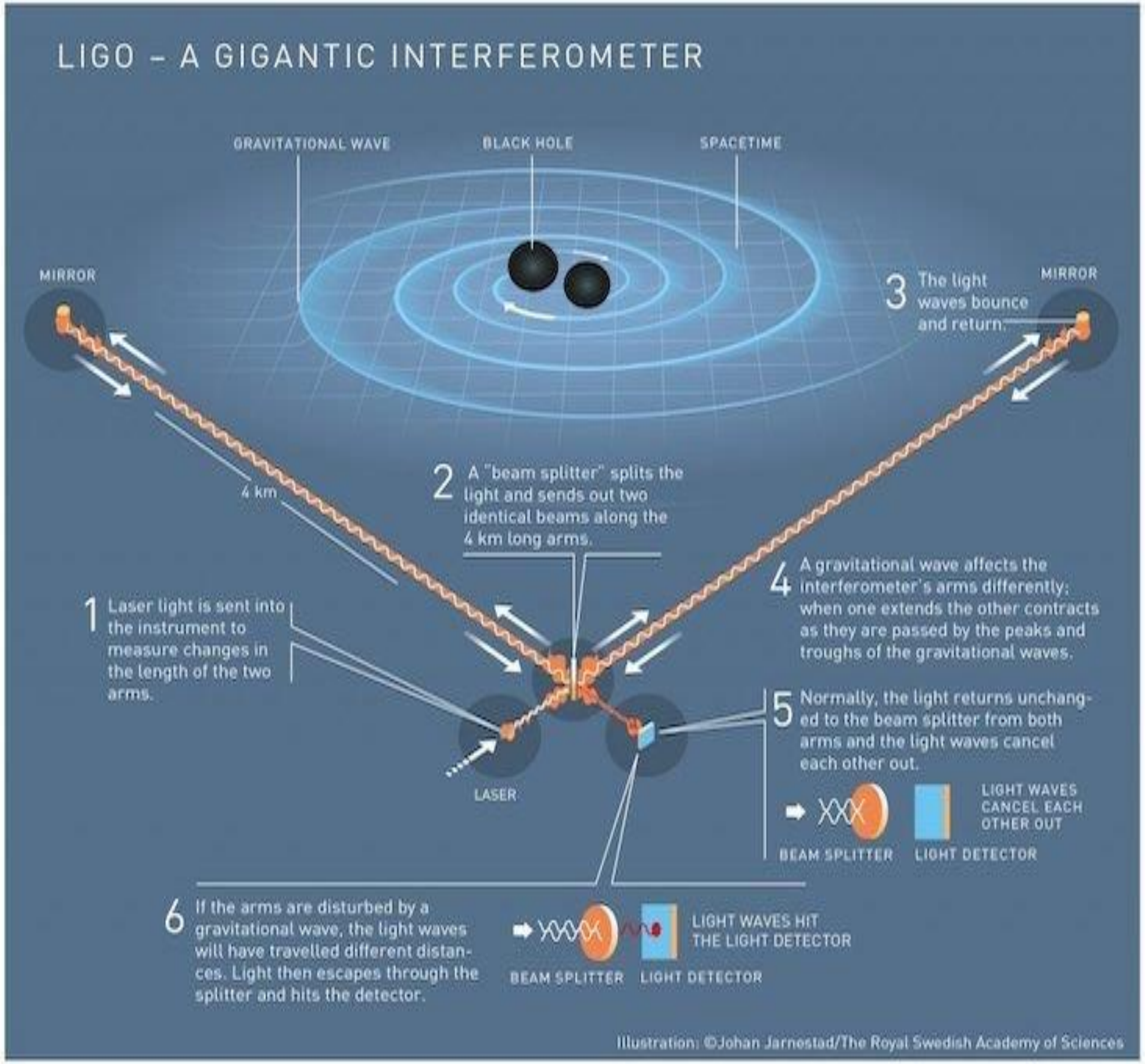
photon **phonon** **interaction**

- Optomechanical interaction leads to uncertainties in interferometric measurement
- But we can also exploit this **to control the behavior of mechanical resonators or the behavior of light**.

Cavity Optomechanics: Bring Mechanical Vibrations to Quantum Regime

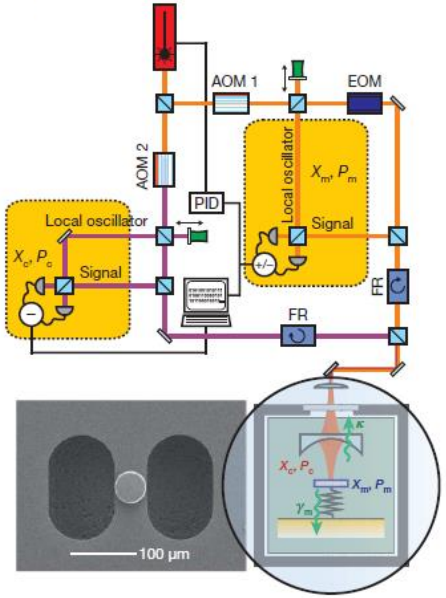
LIGO: **L**aser **I**nterferometer **G**ravitational-Wave **O**bservatory

Cavity Optomechanics: Bring Mechanical Vibrations to Quantum Regime

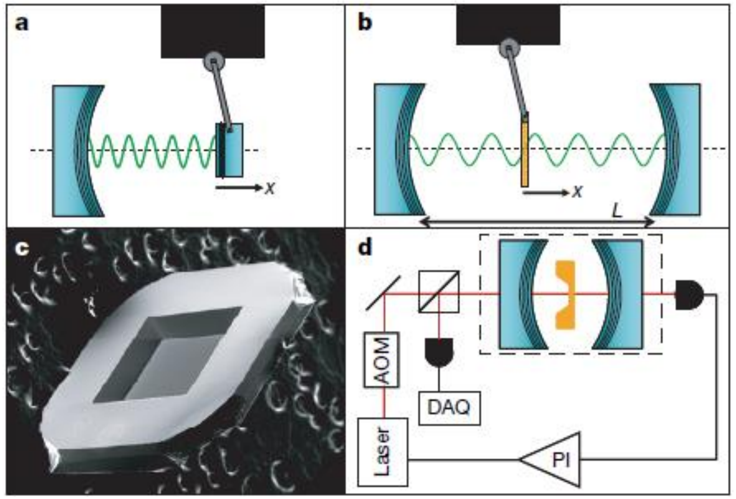


- Noise from the light source (e.g. shot noise)
- "Back-action noise" originating from mechanically perturbed mirror due to the radiation pressure of the light.
- The precision of the measurement is limited by the intensity of the light an d the back-action noise.
- Optimal intensity of light compromisi ng the two effects needed to be find.

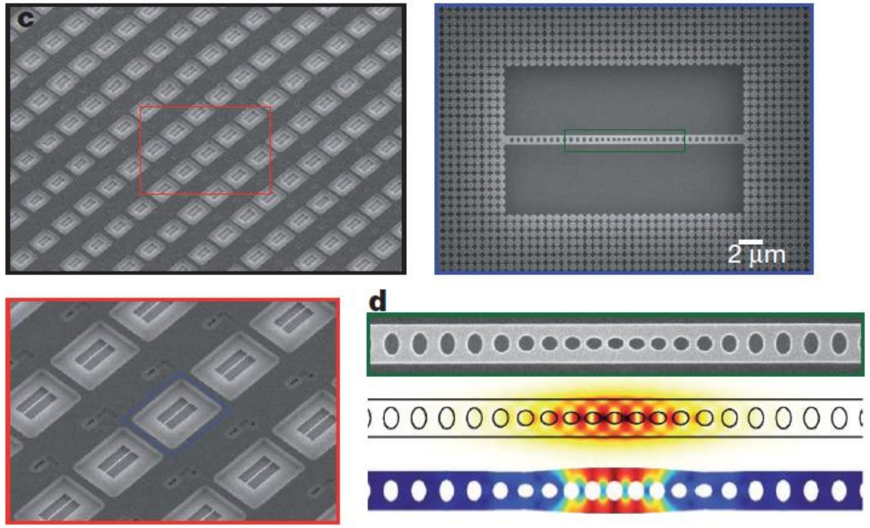
Cavity Optomechanics: Bring Mechanical Vibrations to Quantum Regime



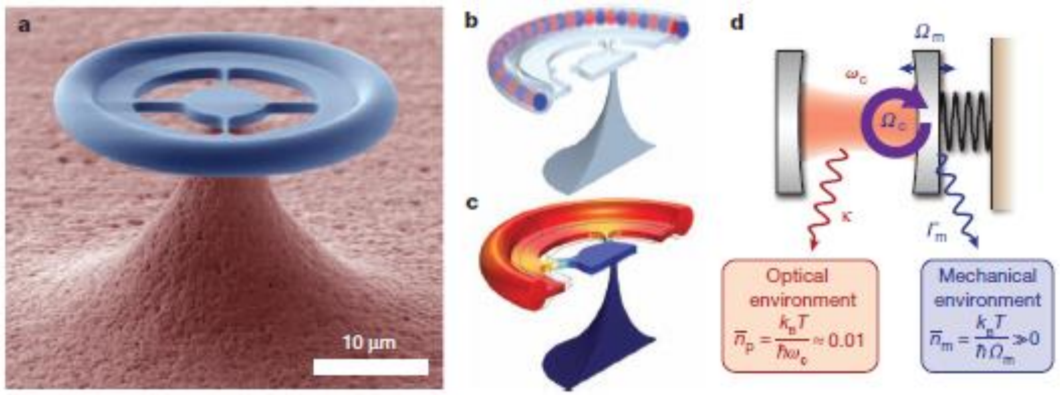
Nature ***460***, 724 – 727 (2009)



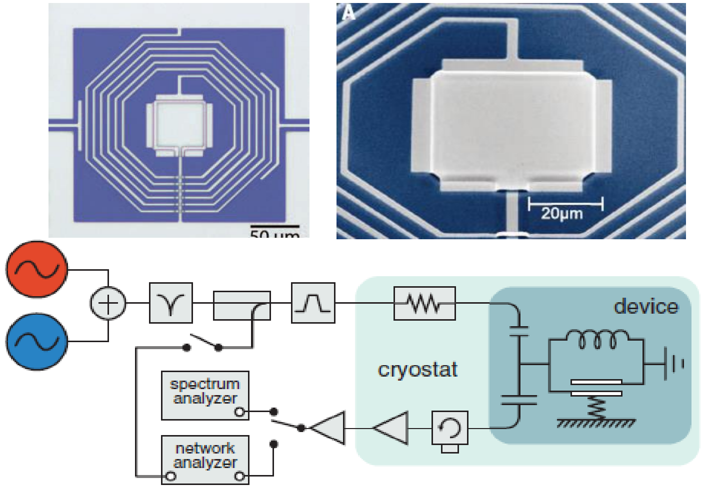
Nature ***452***, 72 – 75 (2008)



Nature ***472***, 69 – 73 (2011)

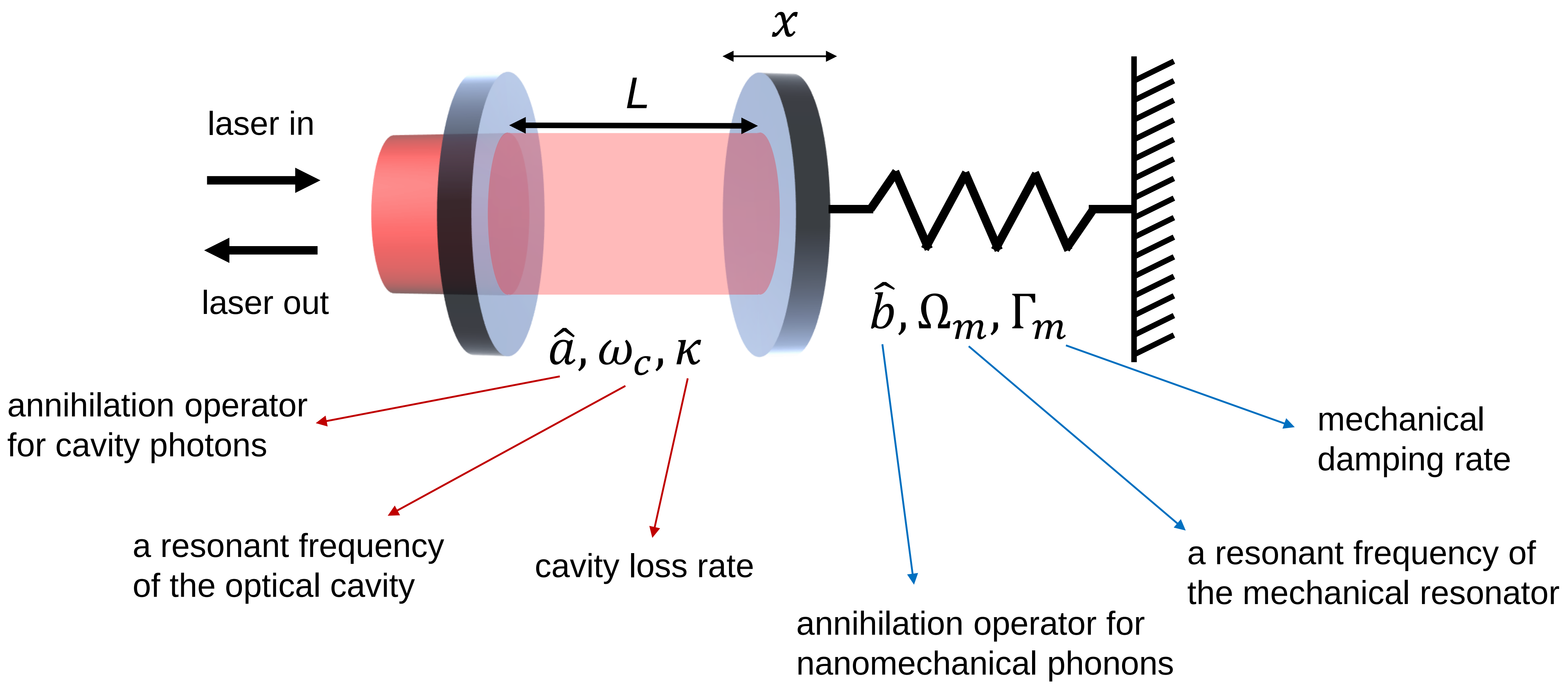


Nature ***482***, 63 – 67 (2012)

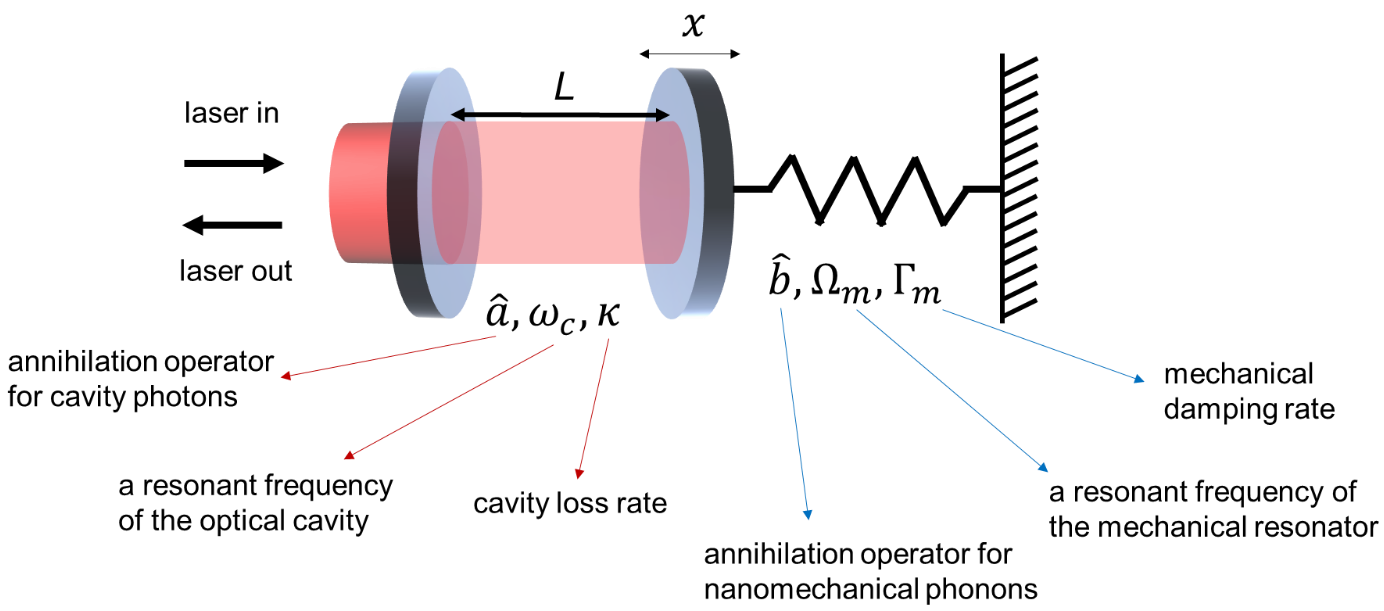


Science 349, 952 – 955 (2015)

Introduction to Cavity Optomechanics (Some Mathematics)



Introduction to Cavity Optomechanics (Some Mathematics)



Based on the formalism we discussed so far in the theory section, we begin with a Hamiltonian

$$\hat{H} = \hbar\omega_c(x)\hat{a}^\dagger\hat{a} + \hbar\Omega_m\hat{b}^\dagger\hat{b}$$

Here, we note that the resonant frequency of the cavity depends on the displacement of the mechanical resonator.

If we approximate $\omega_c(x)$ using the Taylor's expansion up to the first order, we will have

$$\omega_c(x) = \omega_c(x=0) + \frac{d\omega_c}{dx}x$$

A resonant frequency of the optical cavity when there is no mechanical displacement

This term describes the change of the optical cavity frequency when there is a mechanical displacement

Introduction to Cavity Optomechanics (Some Mathematics)

If we insert the approximation to the Hamiltonian equation, we get

$$\hat{H} = \hbar\omega_c(x=0)\hat{a}^\dagger\hat{a} + \hbar\Omega_m\hat{b}^\dagger\hat{b} + \hbar\frac{d\omega_c}{dx}x\hat{a}^\dagger\hat{a}$$

If we describe the displacement in terms of the creation ($\hat{b}^\dagger$) and annihilation operators ($\hat{b}$) of phonon, we can express x as in the following:

$$x = x_{zpf}(\hat{b}^\dagger + \hat{b}) \quad \text{with} \quad x_{zpf} = \sqrt{\frac{\hbar}{2m\omega}}$$

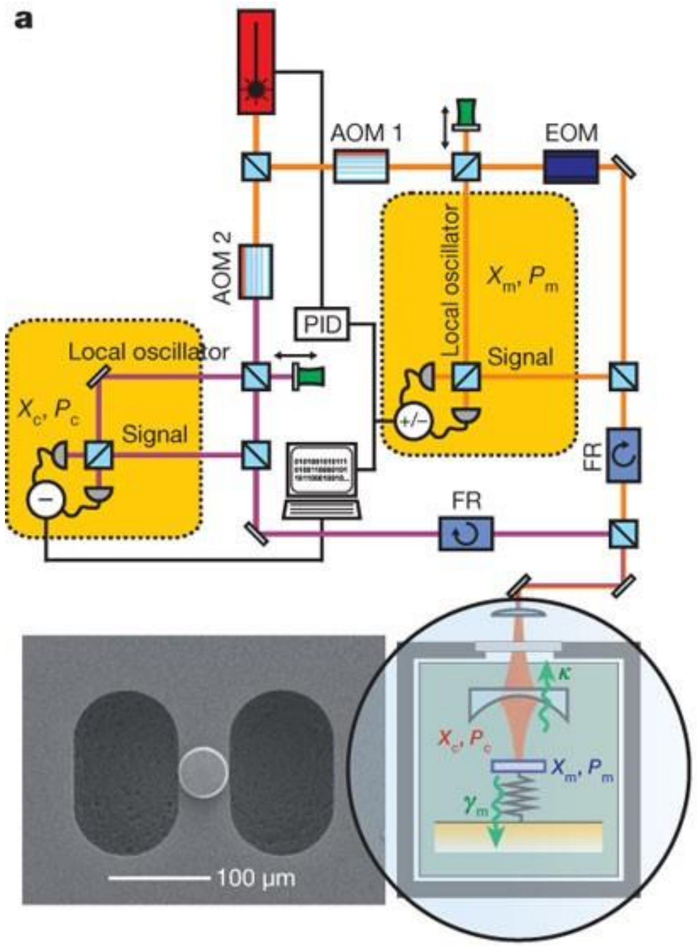
If we use this formula, the Hamiltonian then becomes

$$\hat{H} = \hbar\omega_c(x=0)\hat{a}^\dagger\hat{a} + \hbar\Omega_m\hat{b}^\dagger\hat{b} + \hbar\frac{d\omega_c}{dx}x_{zpf}(\hat{b}^\dagger + \hat{b})\hat{a}^\dagger\hat{a}$$

vacuum optomechanical coupling rate

Introduction to Cavity Optomechanics (Some Mathematics)

Example) Calculate the single-photon optomechanical coupling of an optical cavity-mechanical resonator system shown below. Parameters are given in the following.



Nature ***460***, 724 – 727 (2009)

Parameters:

$$m = 145\, ng$$
$$\Omega_m = 2\pi \times 947\, kHz$$
$$L = 25\, mm$$
$$\omega_c = 2\pi \times 282\, THz\ (\sim 1064\, nm)$$

Solution) For Fabry-Perot cavity,

Taylor's expansion

$$\omega_c(x) = \frac{L}{L+x}\omega_{c,x=0} = \left(1 + \frac{x}{L}\right)^{-1} \omega_{c,x=0} \approx \left(1 - \frac{x}{L}\right)\omega_{c,x=0}$$

$$g_0 = -\frac{\partial \omega_c}{\partial x} x_{zpf} = \frac{\omega_{c,x=0}}{L}\sqrt{\frac{\hbar}{2m\Omega_m}} = 2\pi \times 2.79\, Hz$$

Introduction to Cavity Optomechanics (Some Mathematics)

If we let $g_0 = -\frac{d\omega_c}{dx} x_{zpf}$, and write the Hamiltonian considering an external drive field (note that the optomechanical interaction can only be realized when there is an external drive), we reach

$$\hat{H} = \hbar\omega_c(x=0)\hat{a}^\dagger\hat{a} + \hbar\Omega_m\hat{b}^\dagger\hat{b} - \hbar g_0(\hat{b}^\dagger + \hat{b})\hat{a}^\dagger\hat{a} + \hat{H}_{drive}$$

optomechanical interaction

with $$\hat{H}_{drive} = i\hbar\alpha_{in}\sqrt{\kappa_{ext}}(\hat{a}^\dagger e^{-i\omega_d t} + \hat{a}e^{i\omega_d t})$$

Laser intensity (α_{in}) coupling rate to cavity (κ_{ext}) laser drive frequency (ω_d)

To remove the time-dependent term, we consider the Hamiltonian in a new frame rotating at the drive laser frequency ω_d, by applying the unitary transformation with $\hat{U} = e^{i\omega_d \hat{a}^\dagger \hat{a} t}$. The new Hamiltonian is given by

$$\hat{H}_{new} = \hat{U}\hat{H}\hat{U}^\dagger - i\hbar\hat{U}\partial\hat{U}^\dagger/\partial t$$

Introduction to Cavity Optomechanics (Some Mathematics)

By taking one more approximation called linear approximation $\hat{a} = \sqrt{n_c} + \hat{a}$,

we obtain the optomechanical Hamiltonian as in the following:

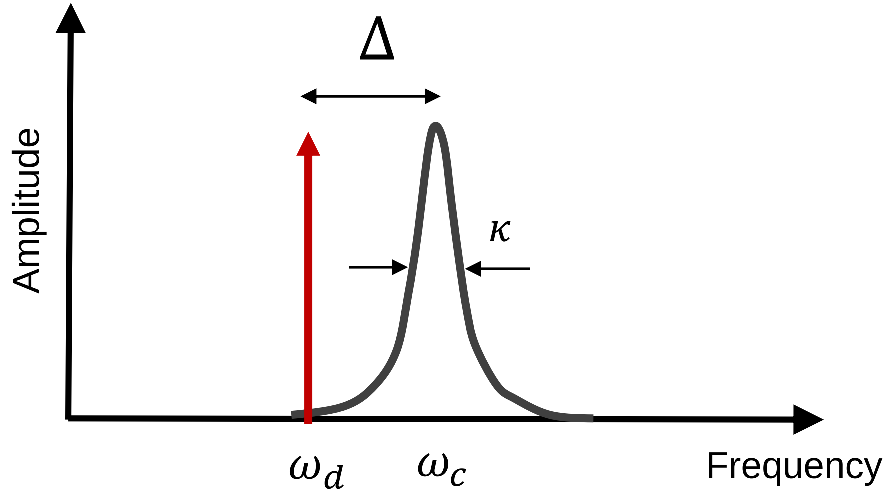
$$\hat{H} = -\hbar\Delta\hat{a}^\dagger\hat{a} + \hbar\Omega_m\hat{b}^\dagger\hat{b} - \hbar g_0\sqrt{n_c}(\hat{a}^\dagger + \hat{a})(\hat{b}^\dagger + \hat{b})$$

- We neglect the driving terms and other small terms for the simplicity

- Here the most important term is detuning $\Delta = \omega_d - \omega_c$, which denotes the difference between the driving frequency and the cavity frequency.

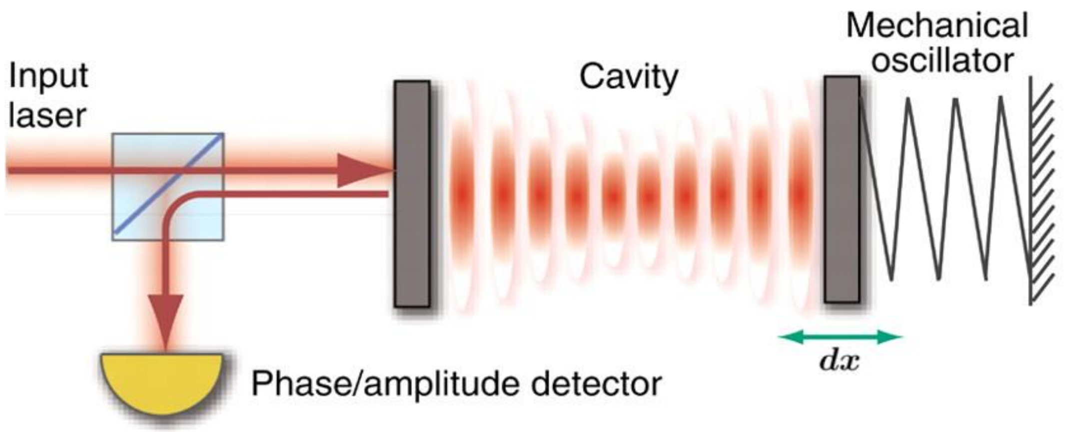
- Depending on the detuning, optomechanical systems exhibit different behaviors and we will see in the following.

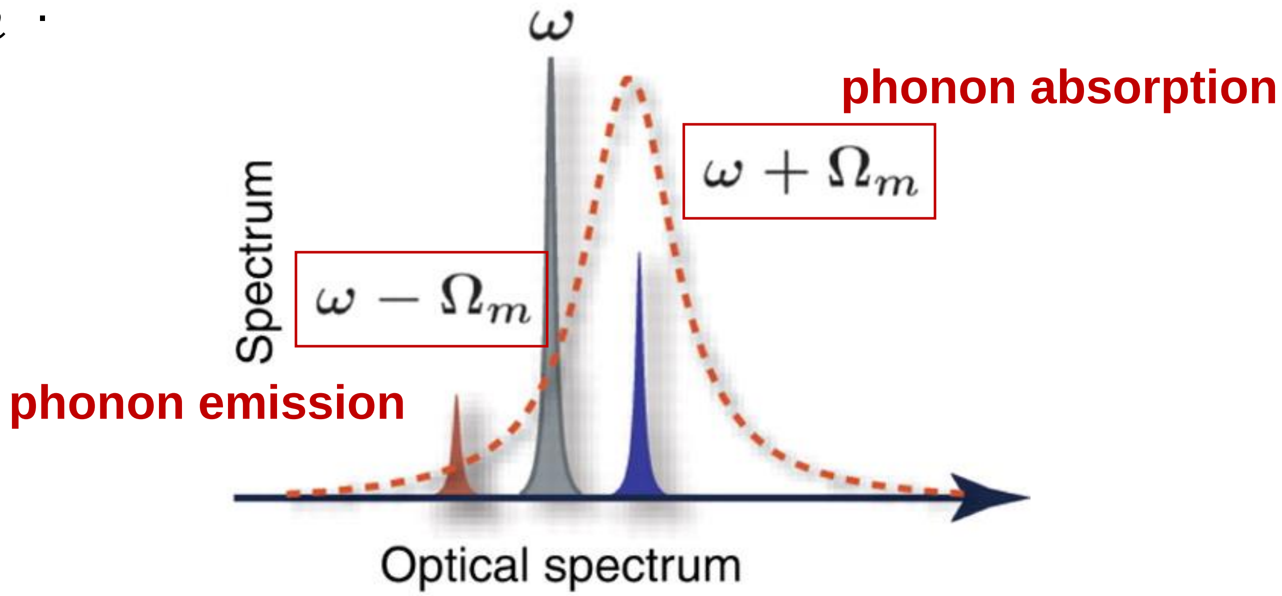
- Here, $g = g_0\sqrt{n_c}$ is general optomechanical coupling rate. This means that the coupling depends on the strength of the drive field.



Introduction to Cavity Optomechanics

Let's consider a Fabry-Perot cavity in the following figure. If the cavity is modulated by the mechanical motion, the optical responses we measure using the detector shows the laser intensity oscillation at the frequency of the mechanical resonator Ω_m. If we consider this modulation process in the optical spectrum domain, we can see sidebands generated at $\omega_d + \Omega_m$ and $\omega_d - \Omega_m$.





T. J. Kippenberg, Cavity Optomechanics: Back-Action at the Mesoscale, ***Science*** 321, 1172 - 1176 (2008)

In cavity optomechanics, we manipulate this sideband generation process to achieve desired responses of systems.

Introduction to Cavity Optomechanics (Optomechanical Cooling)

One of the representative phenomena we can realize using the optomechanical interaction is **optomechanical cooling.**

When $\Delta = \omega_d - \omega_c = -\Omega_m$ and $\Omega_m \gg \kappa$, optomechanical interaction creates a frequency sideband only around the cavity frequency. This single sideband generation is related to **phonon-absorption process**. This means that we **reduce(or cool down)** the energy of the mechanical resonator using optomechanical interaction.

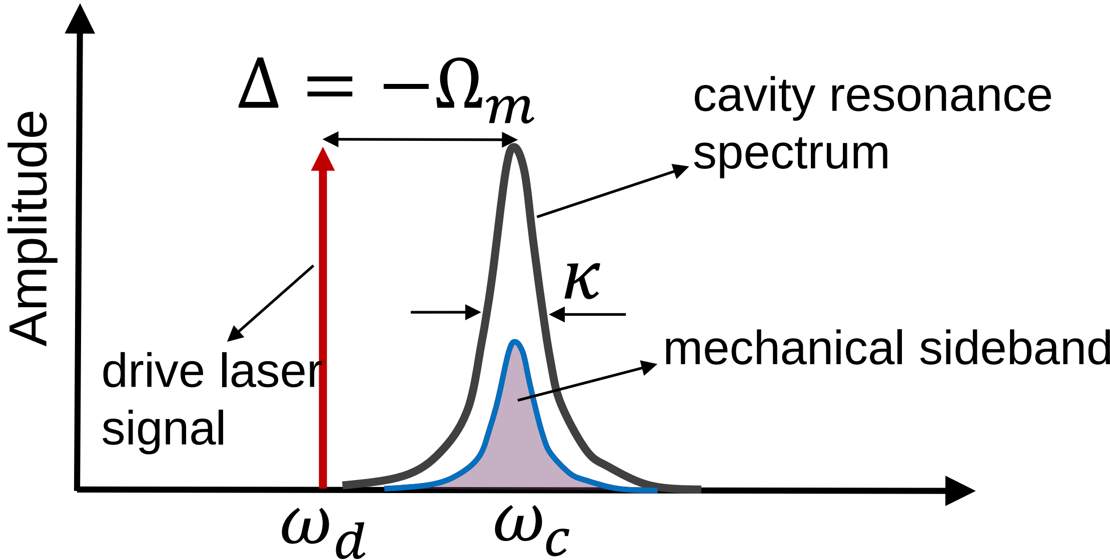
The Hamiltonian then becomes

remove phonon

$$\hat{H} = -\hbar\Delta\hat{a}^\dagger\hat{a} + \hbar\Omega_m\hat{b}^\dagger\hat{b} - \hbar g_0\sqrt{n_c}\left(\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger\right)$$

create cavity photon

This regime is called **red-detuned regime** as the driving laser frequency is smaller and this operation leads to cooling of ‘hot’ thermal phonons to ‘cold’ cavity photons.



Introduction to Cavity Optomechanics (Optomechanical Amplification)

Another representative phenomena is **optomechanical amplification.**

When $\Delta = \omega_d - \omega_c = \Omega_m$ and $\Omega_m \gg \kappa$, optomechanical interaction creates a frequency sideband only around the cavity frequency. This single sideband generation is related to **phonon-creation process**. This means that we **increase(or amplify)** the energy of the mechanical resonator using optomechanical interaction.

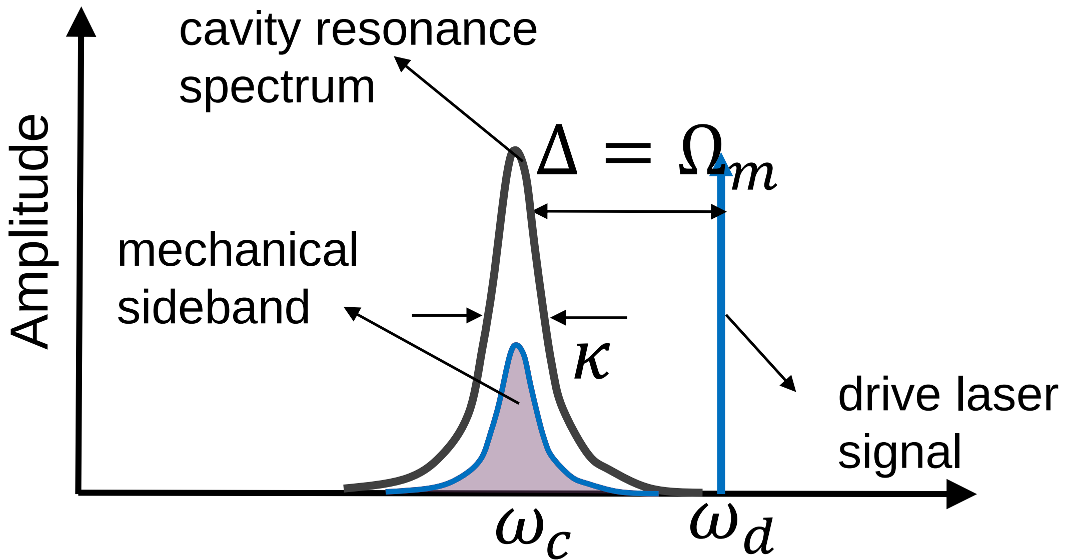
The Hamiltonian then becomes

create phonon

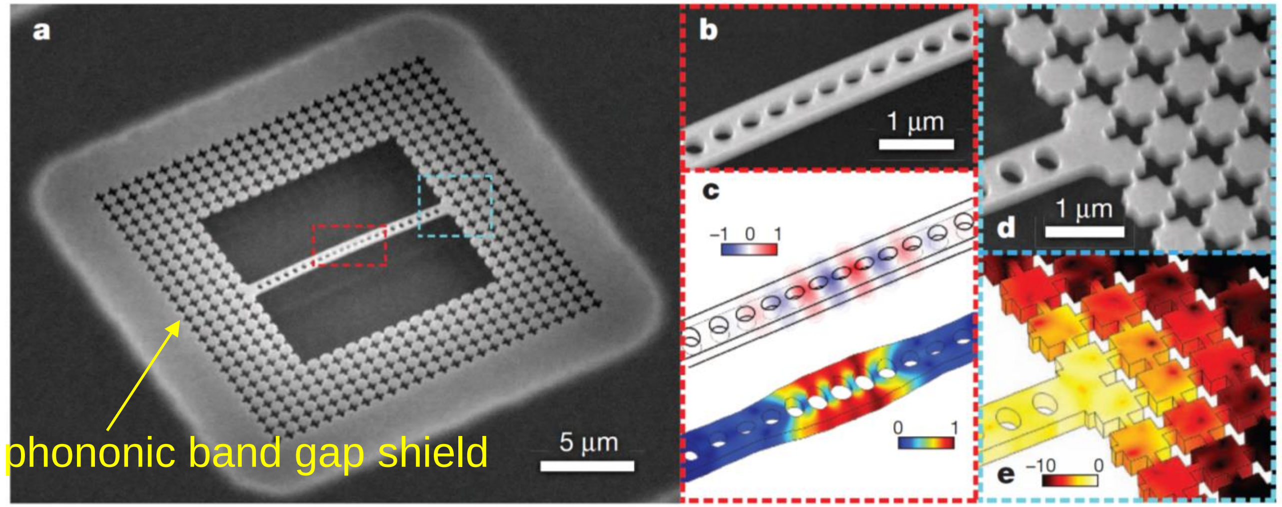
$$\hat{H} = -\hbar\Delta\hat{a}^\dagger\hat{a} + \hbar\Omega_m\hat{b}^\dagger\hat{b} - \hbar g_0\sqrt{n_c}(\hat{a}^\dagger\hat{b}^\dagger + \hat{a}\hat{b})$$

create cavity photon

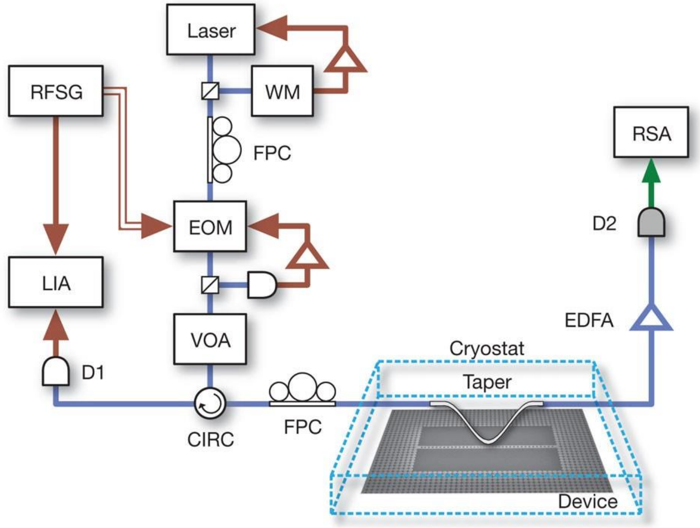
This regime is called **blue-detuned regime** as the driving laser frequency is larger and this operation leads to amplification of phonons.

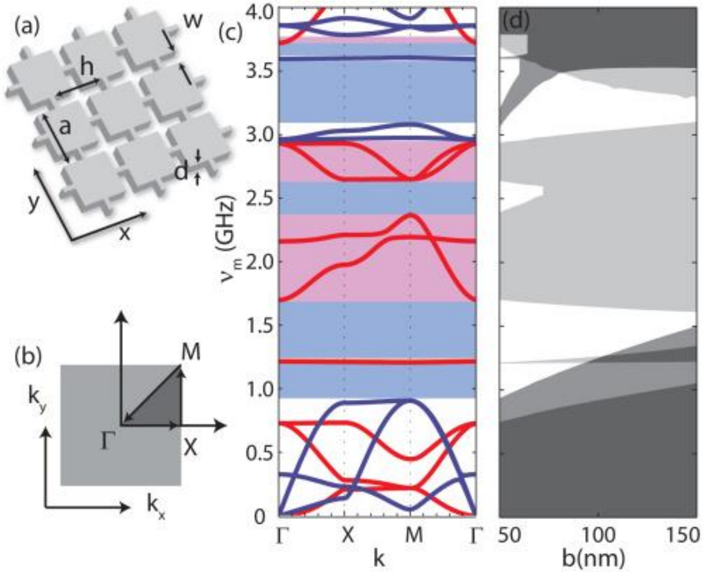


Research Example of Cavity Optomechanics



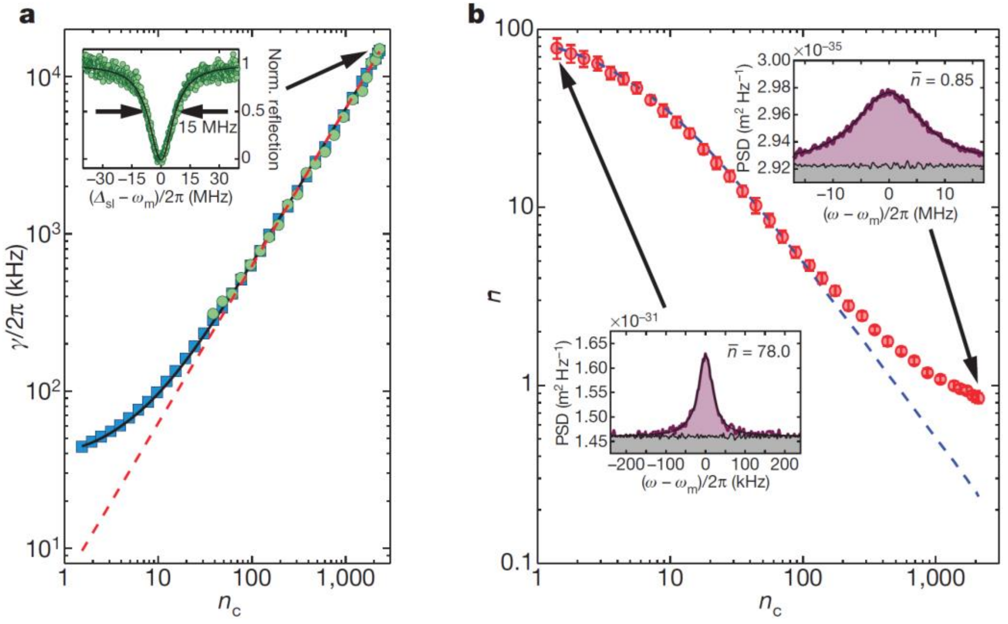
radiation loss to the bulk is suppressed!





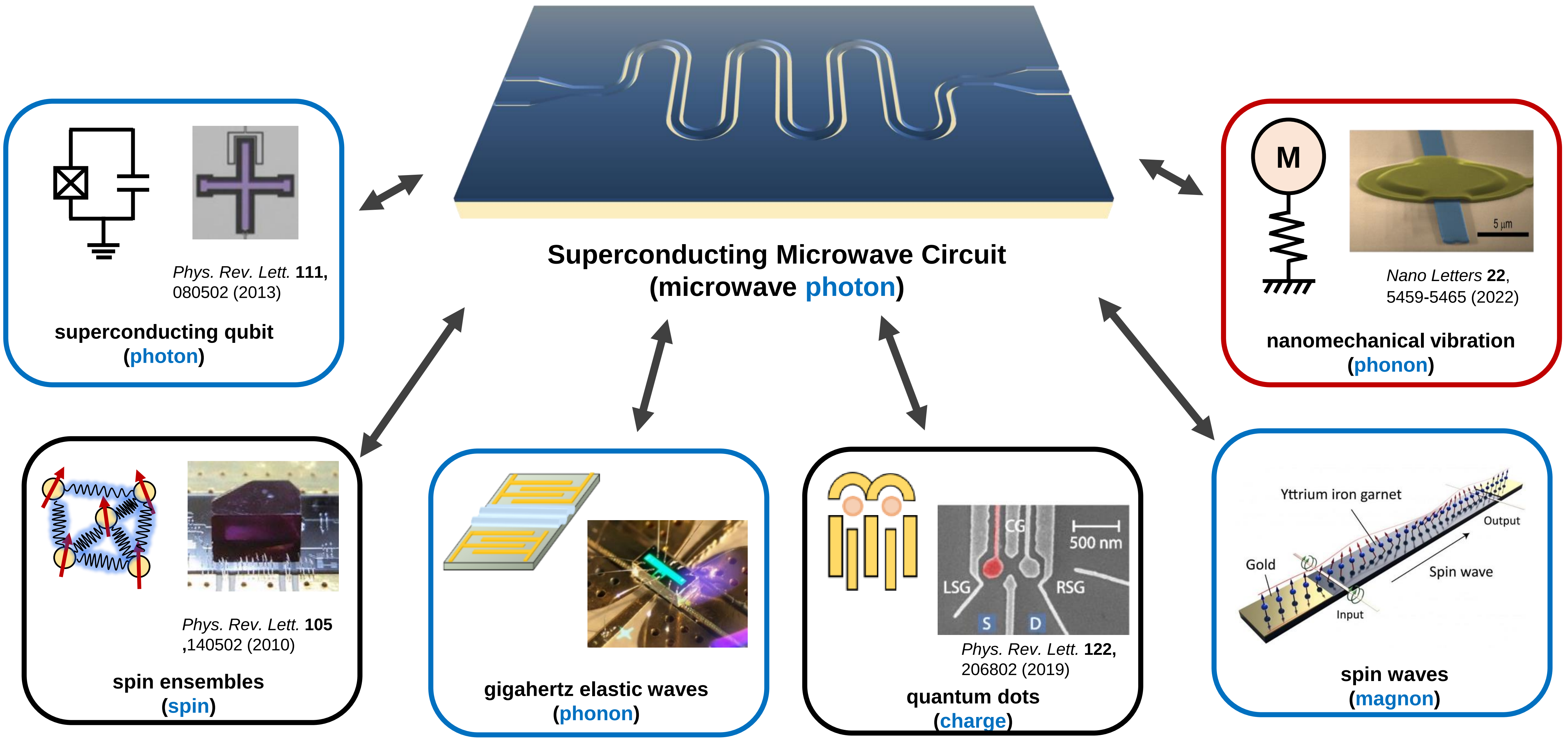
engineering phononic band gap for high-Q resonator

Opt. Express **19**, 5658-5669 (2011)

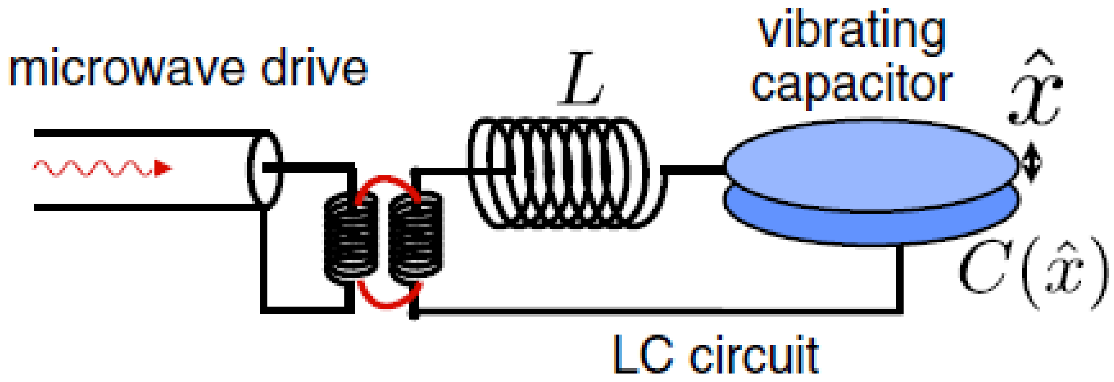


Laser cooling of a nanomechanical oscillator into its quantum ground state.
Nature ***478***, 89 – 92 (2011)

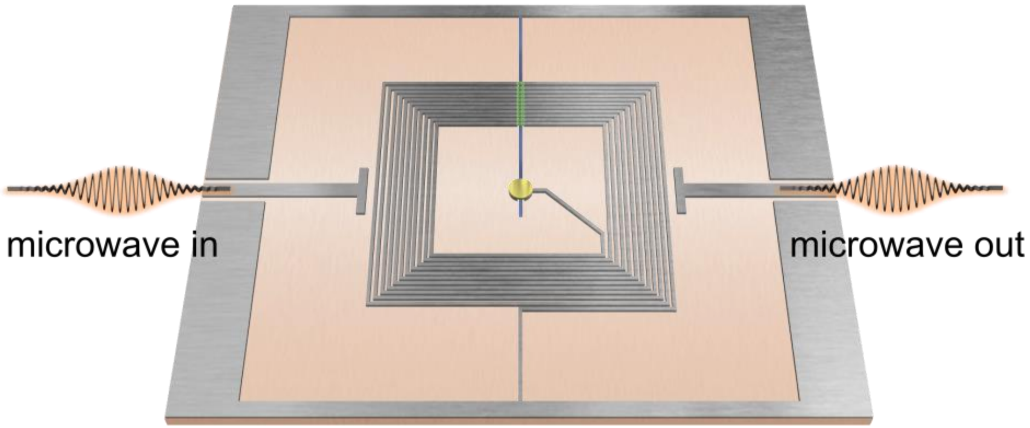
Hybrid Quantum Systems with Superconducting Microwave Circuits



Cavity Optomechanics with Superconducting Microwave Circuits



Review of Modern Physics **86**, 1391 - 1452 (2014)



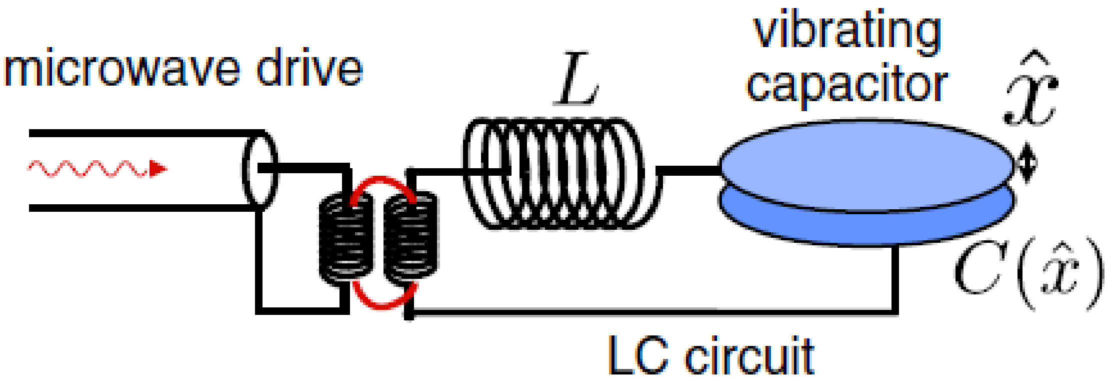
- Superconducting nanoelectromechanical systems can realize cavity optomechanical interaction via microwave fields at GHz frequencies.

- The reason why we are using **superconducting materials** for such devices is that **microwave loss properties can extremely be enhanced** as a superconducting material has zero electrical resistivity below its superconducting temperature.

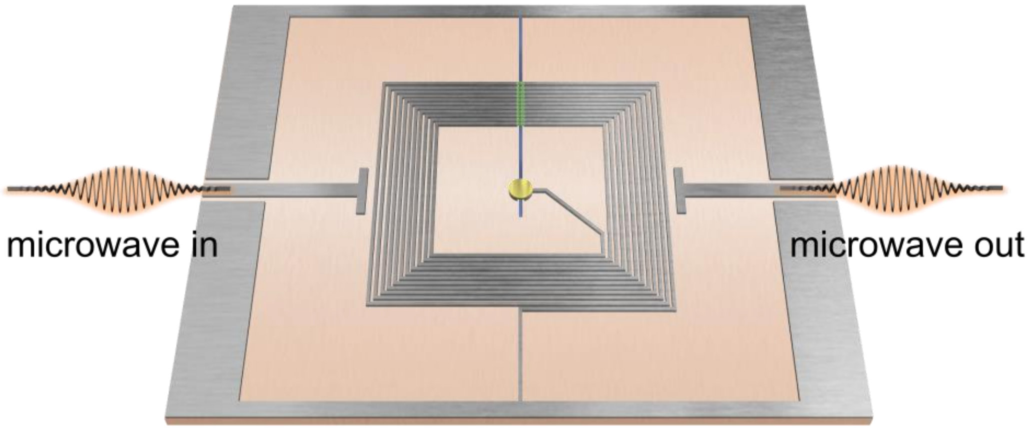
- Furthermore, as we discussed, mK environments enable the quantum ground state of GHz microwave resonators.

- The system can easily be modelled using LC circuit where the **capacitance depends on the mechanical displacement**. Their coupling is realized by **electromechanical interaction** where the voltage applied to the capacitor leads to mechanical displacement via **electrostatic interactions**.

Cavity Optomechanics with Superconducting Microwave Circuits



Review of Modern Physics **86**, 1391 - 1452 (2014)



- The Hamiltonian describing this system is given by

$$\hat{H} = \underbrace{\frac{1}{2}LI^2 + \frac{1}{2}C(x)V^2}_{\text{microwave resonator}} + \underbrace{\frac{p^2}{2m} + \frac{1}{2}m\Omega_m^2 x^2}_{\text{mechanical resonator}}$$

$$C(x) = \frac{\varepsilon_0 A}{(d+x)}$$ for parallel capacitor

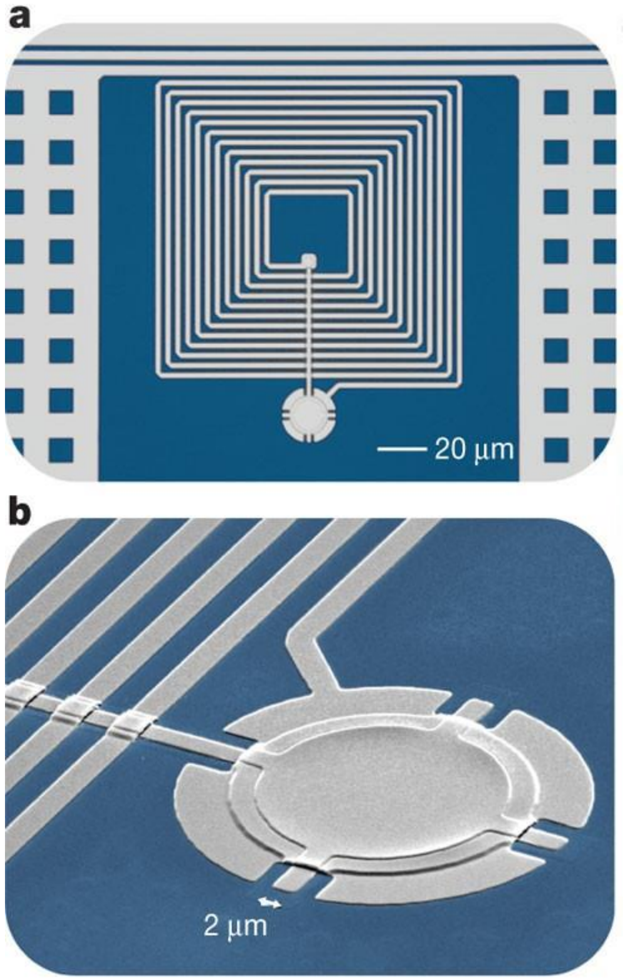
- The Hamiltonian describes two-coupled harmonic oscillators. If we express the Hamiltonian quantum mechanically, we get

$$\hat{H} = \underbrace{\hbar\omega_c \hat{a}^\dagger \hat{a}}_{\textbf{photon}} + \underbrace{\hbar\Omega_m \hat{b}^\dagger \hat{b}}_{\textbf{phonon}} - \underbrace{\hbar g_0 \hat{a}^\dagger \hat{a}\left(\hat{b}^\dagger + \hat{b}\right)}_{\textbf{interaction}}$$

$$g_0 = -\frac{\partial \omega_c}{\partial x} x_{zpf}$$ single photon optomechanical coupling constant

Cavity Optomechanics with Superconducting Microwave Circuits

Example) Calculate the single-photon optomechanical coupling of a superconducting nanoelectromechanical device shown in the below. Parameters are given in the following.



J.D. Teufel, *et al. Nature* **471**, 204-208(2011)

Parameters: $t = 100\ nm$; $D = 15\ \mu m$

$$\rho_{Al} = 2700\,\frac{kg}{m^3};\quad \Omega_m = 2\pi \times 10.69\ MHz$$

$$d_{cap} = 50\ nm$$

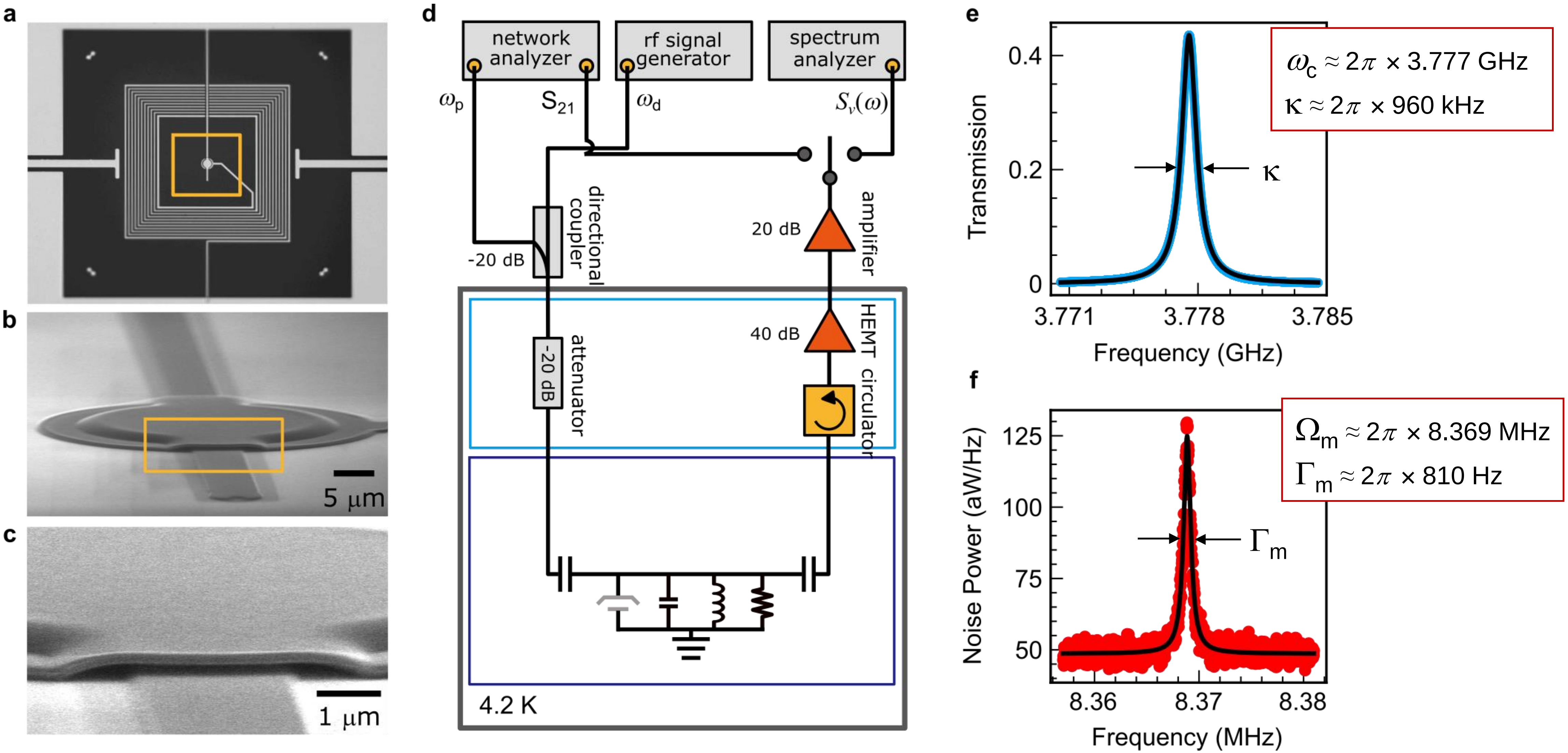
$$L = 12\ nH;\quad C_{x=0} = 38\ fF$$

Solution) For LC circuit, the resonant frequency is

$$\omega_c(x) = \frac{1}{\sqrt{LC(x)}} = \sqrt{\frac{d+x}{LA\varepsilon_0}} = \sqrt{\frac{d\left(1+\frac{x}{d}\right)}{LA\varepsilon_0}} \approx \sqrt{\frac{1}{LC_{x=0}}}\left(1+\frac{x}{2d}\right)$$

$$g_0 = -\frac{\partial \omega_c}{\partial x} x_{zpf} = -\frac{(LC_{x=0})^{-0.5}}{2d}\sqrt{\frac{\hbar}{2m\Omega_m}} \approx 2\pi \times 300\ Hz$$

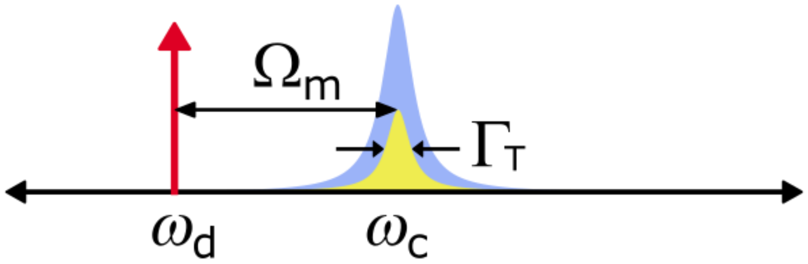
Cavity Optomechanics with Superconducting Microwave Circuits

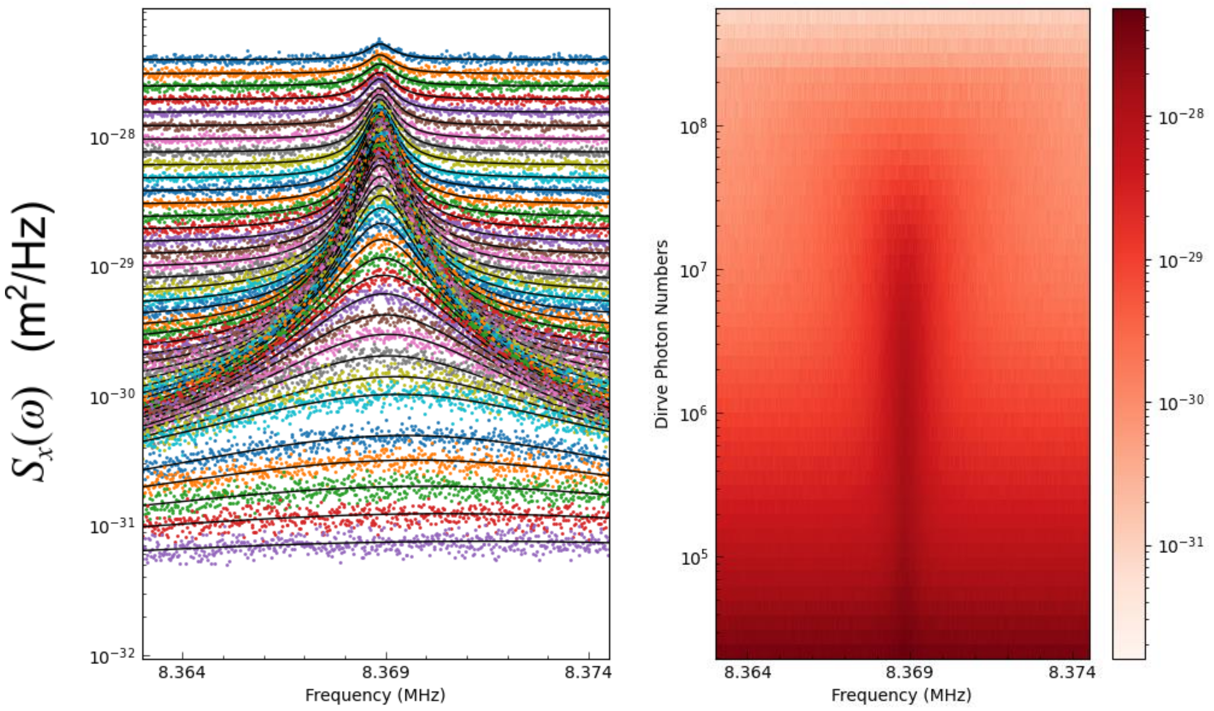


J Cha, *et al.* ***Nano Letters* 21**, 1800-1806 (2021)

Cavity Optomechanics with Superconducting Microwave Circuits

Optomechanical Cooling





- The Hamiltonian is

$$\hat{H} = -\hbar\Delta\hat{a}^\dagger\hat{a} + \hbar\Omega_m\hat{b}^\dagger\hat{b} - \hbar g_0\sqrt{n_d}\left(\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger\right)$$

- **Broadening** of mechanical noise spectrum.
  => optomechanical damping effect.

- **Decrease** of the area of the Lorentzian curve
  => **Reduction** of the energy of the nanomechanical resonator.
  => **Reduction** of the phonon number

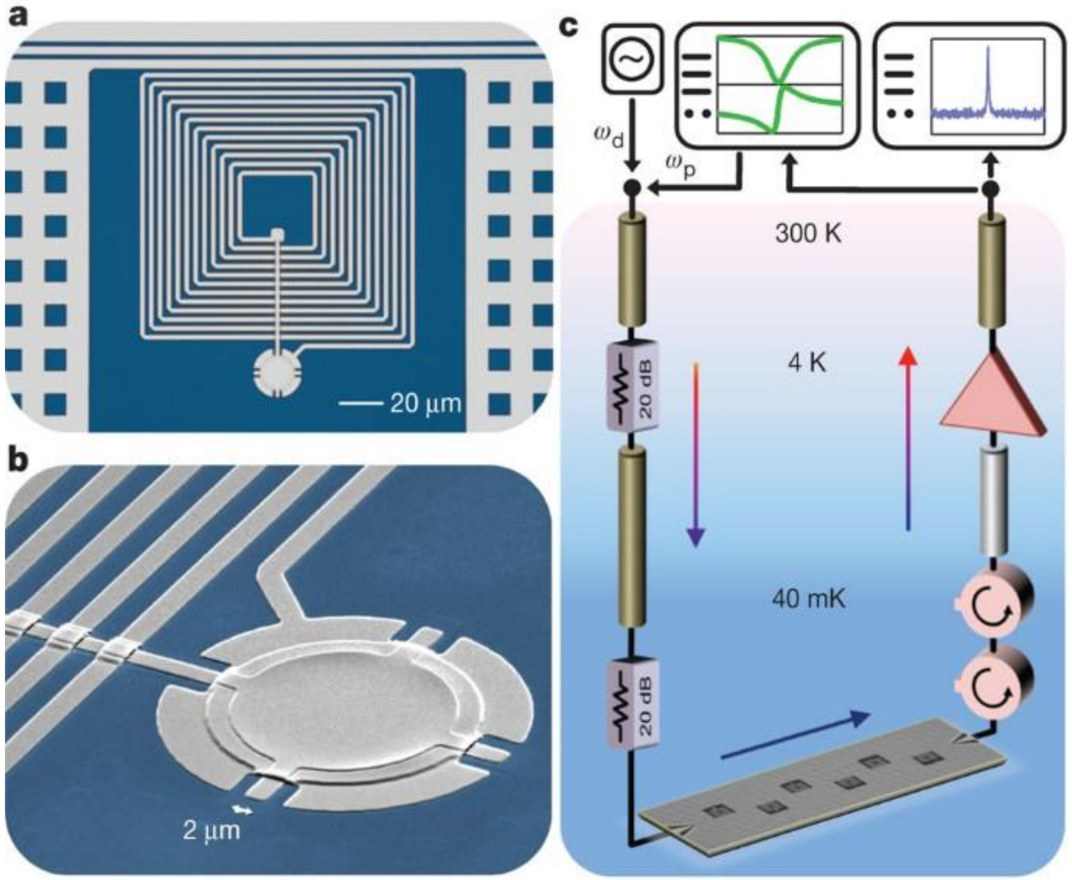
$$A = \int_{-\infty}^{\infty} S_x(\omega)\frac{d\omega}{2\pi} = \langle x^2\rangle = \frac{k_B T}{m\Omega_m^2} = \frac{2k_B T x_{zpf}^2}{\hbar\Omega_\mathrm{m}} = 2n_{ph}x_{zpf}^2$$

$$n_{ph} = \left[\exp\left(\frac{\hbar\Omega_m}{k_B T}\right) - 1\right]^{-1} \approx \frac{k_B T}{\hbar\Omega_m} \quad \text{for} \quad \frac{k_B T}{\hbar\Omega_m} \gg 1$$

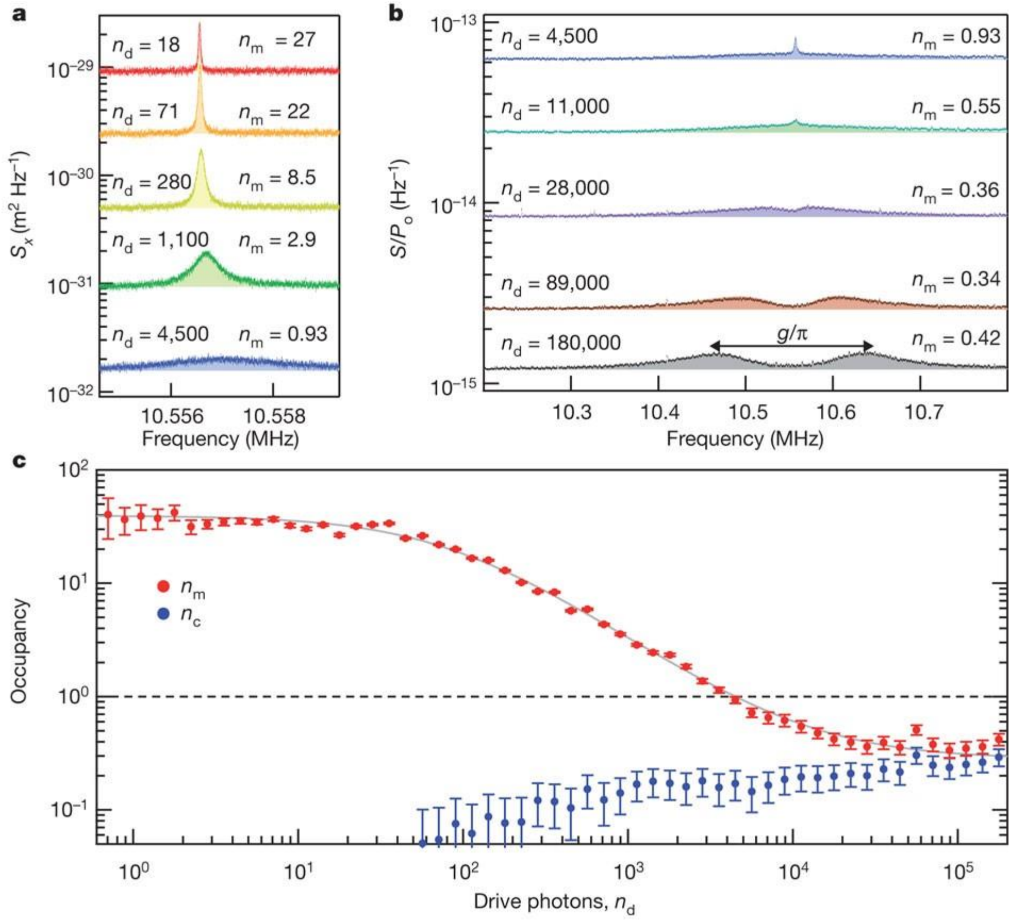
J Cha, *et al. Nano Letters* **21**, 1800-1806 (2021)

Cavity Optomechanics with Superconducting Microwave Circuits

Ground-State Cooling: Quantum Mechanics with Macroscopic Objects



J.D. Teufel, *et al. Nature* **471**, 204-208(2011)

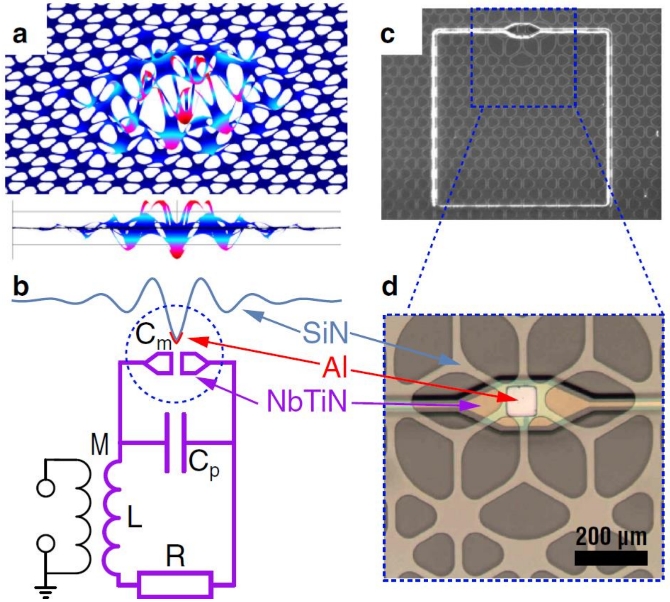


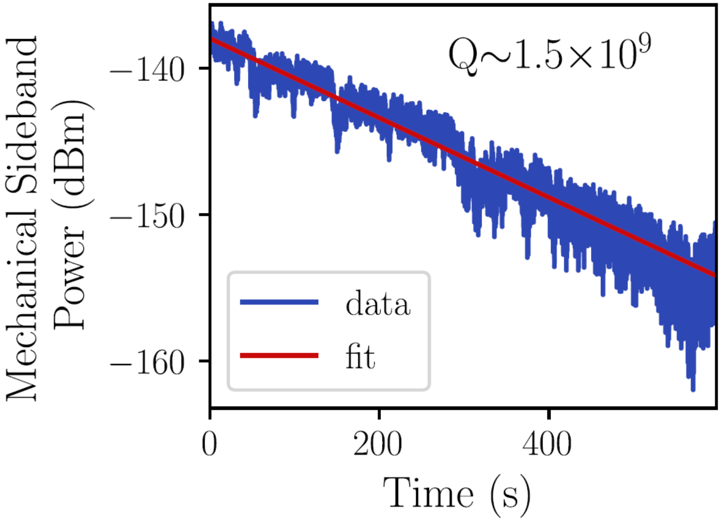
- Ground-state preparation for quantum applications
- Quantum-limited position and force detection

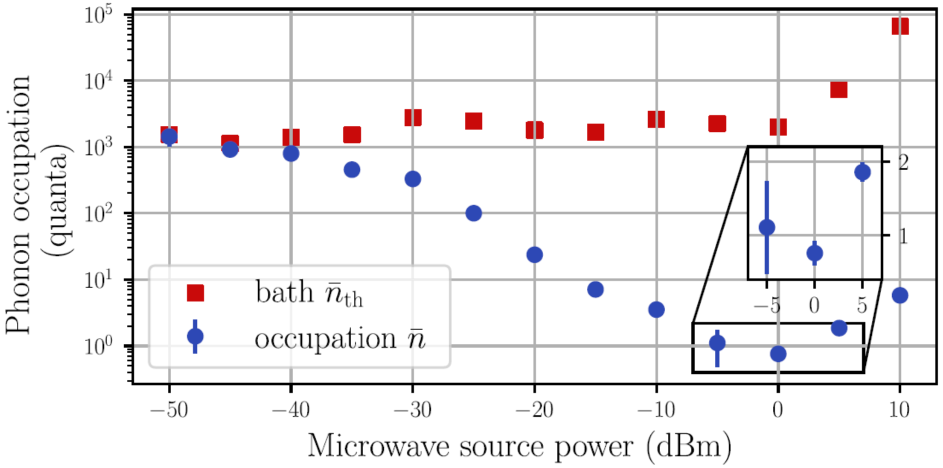
J.D. Teufel, *et al. Nature* **471**, 359-363 (2011)

Cavity Optomechanics with Superconducting Microwave Circuits

Ground-State Cooling with Phononic Crystals



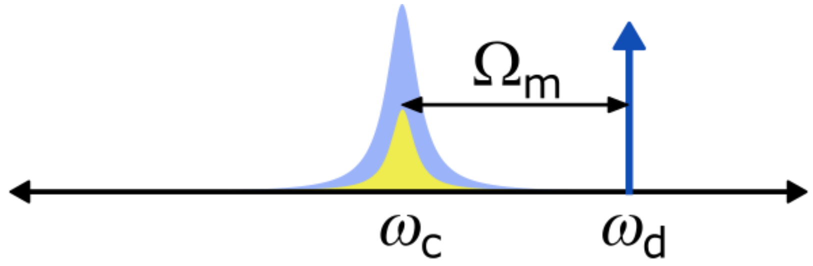


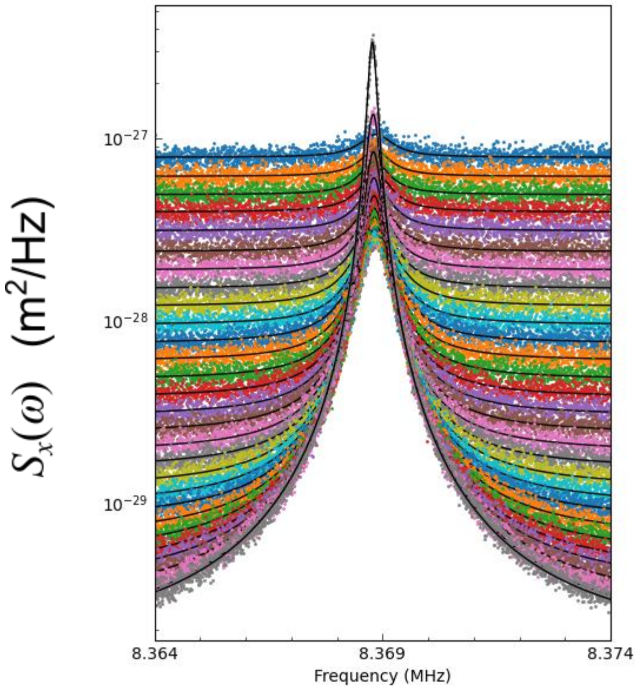


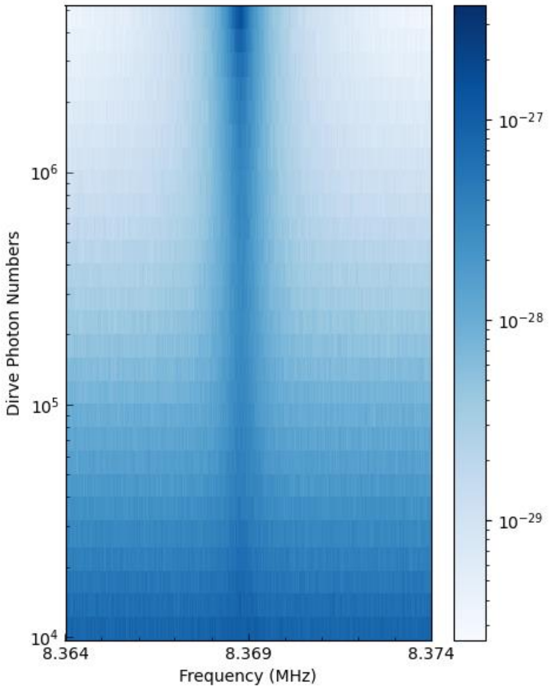
Ground state cooling of an ultracoherent electromechanical system. *Nature Communications* **13**, 1507 (2022)

Cavity Optomechanics with Superconducting Microwave Circuits

Optomechanical Amplification







- The Hamiltonian is

$$\hat{H} = -\hbar\Delta\hat{a}^\dagger\hat{a} + \hbar\Omega_m\hat{b}^\dagger\hat{b} - \hbar g_0\sqrt{n_d}\left(\hat{a}^\dagger\hat{b}^\dagger + \hat{a}\hat{b}\right)$$

- **Narrowing** of mechanical noise spectrum.
  => optomechanical anti-damping effect.

- **Increase** of the area of the Lorentzian curve
  => **Increase** of the energy of the nanomechanical resonator.
  => **Increase** of the phonon number

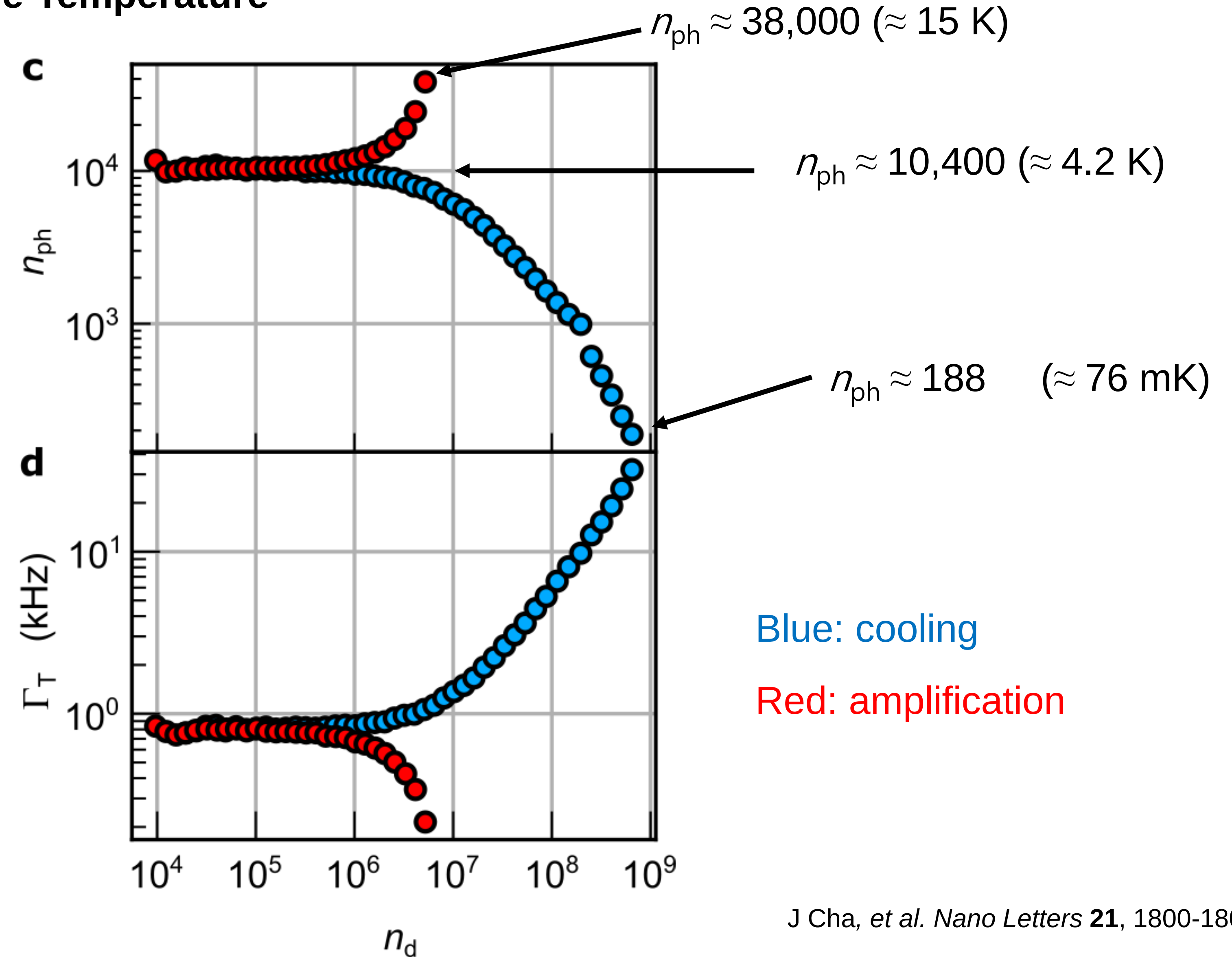
$$A = \int_{-\infty}^{\infty} S_x(\omega)\frac{d\omega}{2\pi} = \langle x^2\rangle = \frac{k_B T}{m\Omega_m^2} = \frac{2k_B T x_{zpf}^2}{\hbar\Omega_\mathrm{m}} = 2n_{ph}x_{zpf}^2$$

$$n_{ph} = \left[\exp\left(\frac{\hbar\Omega_m}{k_B T}\right) - 1\right]^{-1} \approx \frac{k_B T}{\hbar\Omega_m} \qquad \text{for} \quad \frac{k_B T}{\hbar\Omega_m} \gg 1$$

J Cha*, et al. Nano Letters* **21**, 1800-1806 (2021)

Cavity Optomechanics with Superconducting Microwave Circuits

Phonon numbers and Effective Temperature



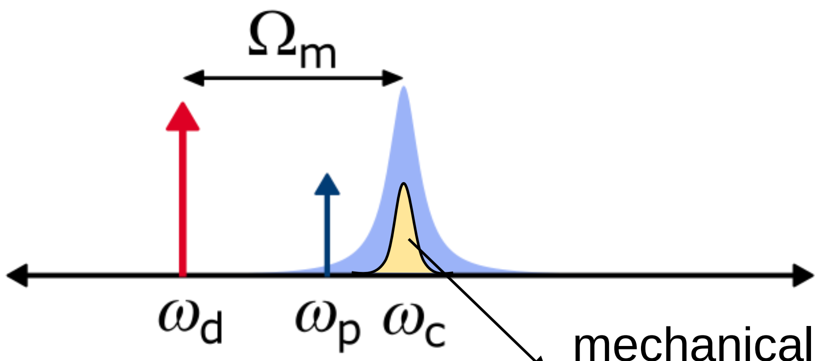
Blue: cooling

Red: amplification

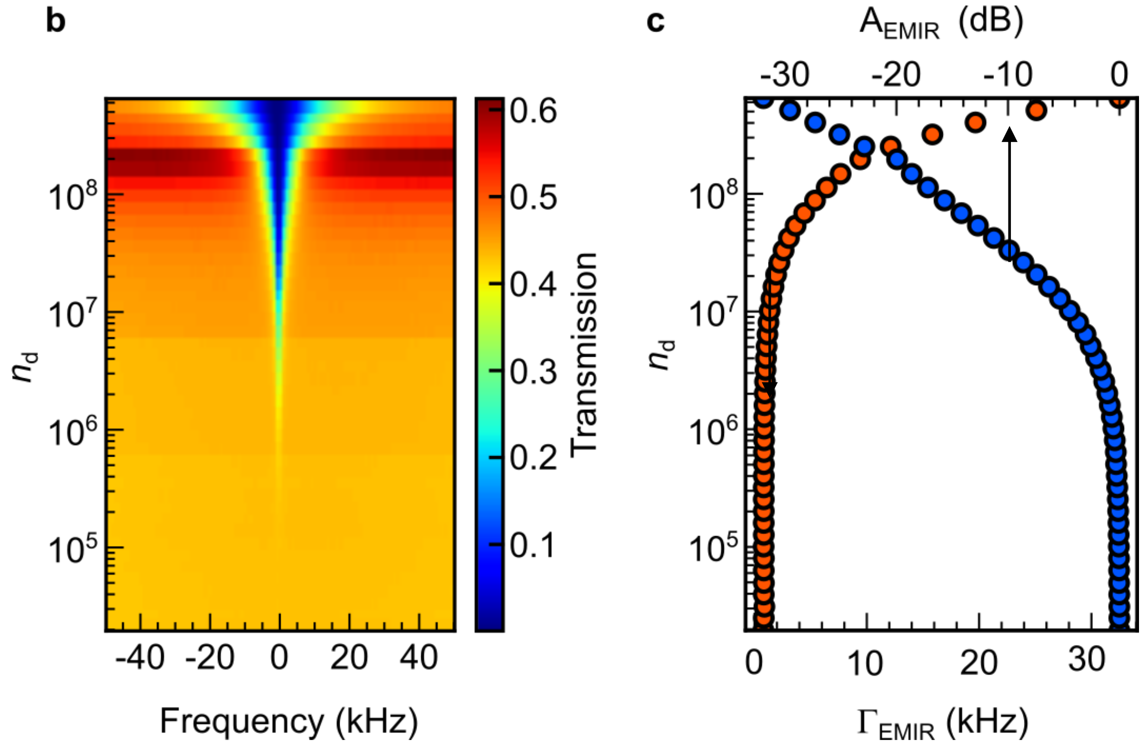
J Cha, *et al. Nano Letters* **21**, 1800-1806 (2021)

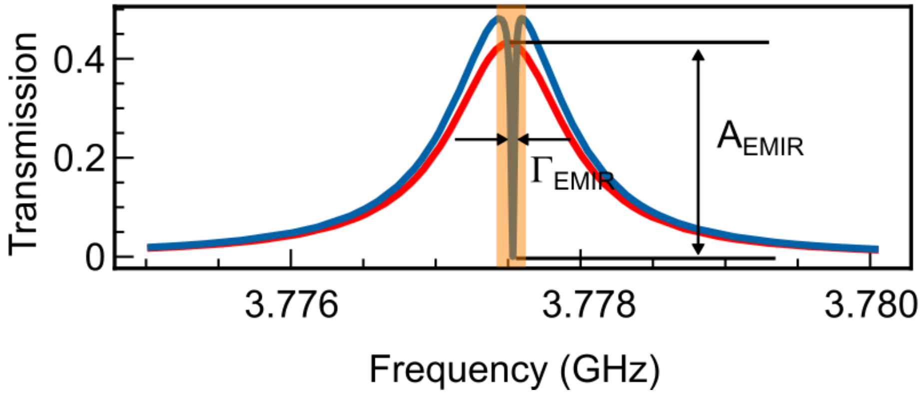
Cavity Optomechanics with Superconducting Microwave Circuits

Optomechanical Control of Microwave Transmission: Optomechanically Induced Transparency



If we send a weak probe beam at ω_p, this probe beam and the mechanical sideband spectrum interfere and change the microwave transmission





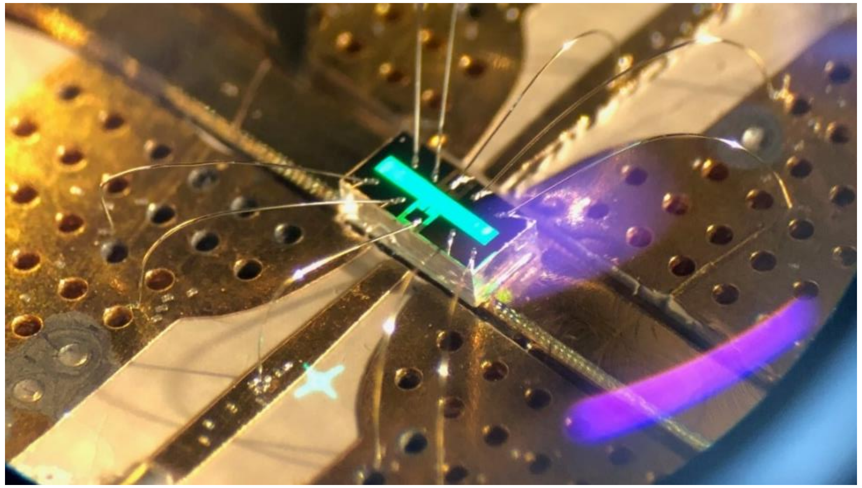
Window linewidth $\Gamma_{\mathrm{EMIR}} = \Gamma_{\mathrm{m}}\left(1 + \frac{4g_0^2 n_d}{\kappa\Gamma_m}\right) = \Gamma_{\mathrm{m}}(1 + C)$

Single photon electromechanical coupling $g_0 \approx 3.3$ Hz

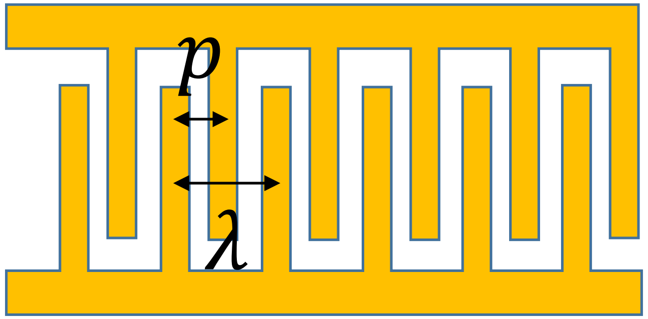
Electromechanical Cooperativity $C \approx 40$

J Cha, *et al. Nano Letters* **21**, 1800-1806 (2021)

High Frequency Acoustic Resonators Coupled to Superconducting Qubits



surface acoustic wave resonators fabricated at KRISS

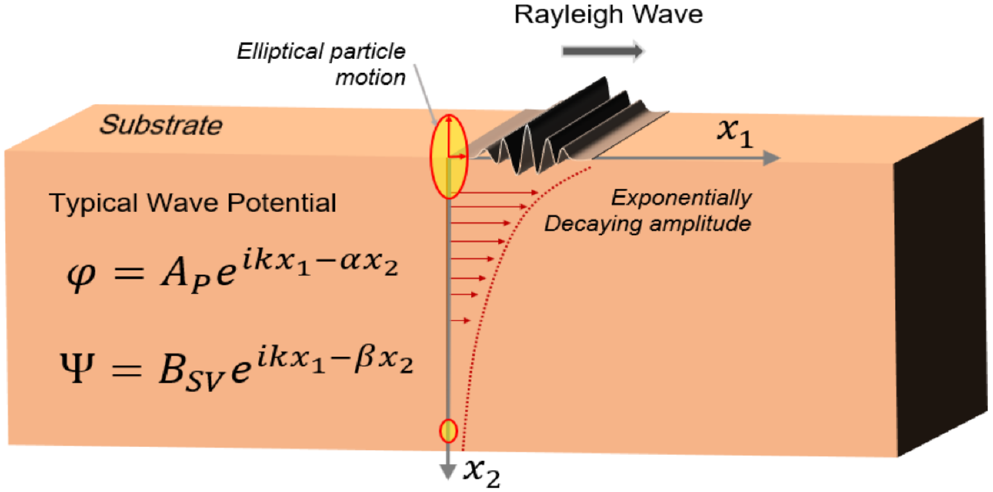


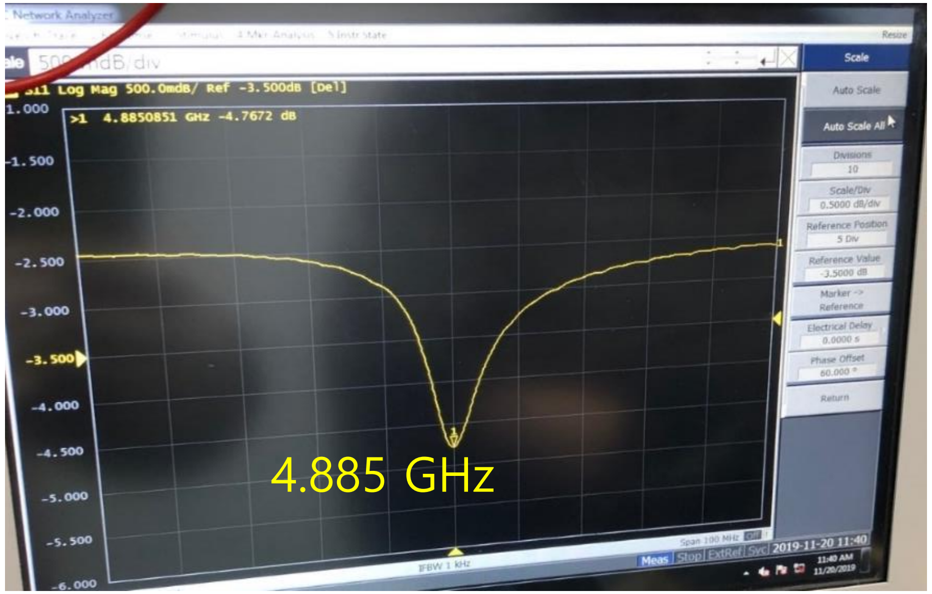
$$f_r = \frac{c}{\lambda} = \frac{c}{2p}$$

c~4000 m/s for $LiNbO_3$

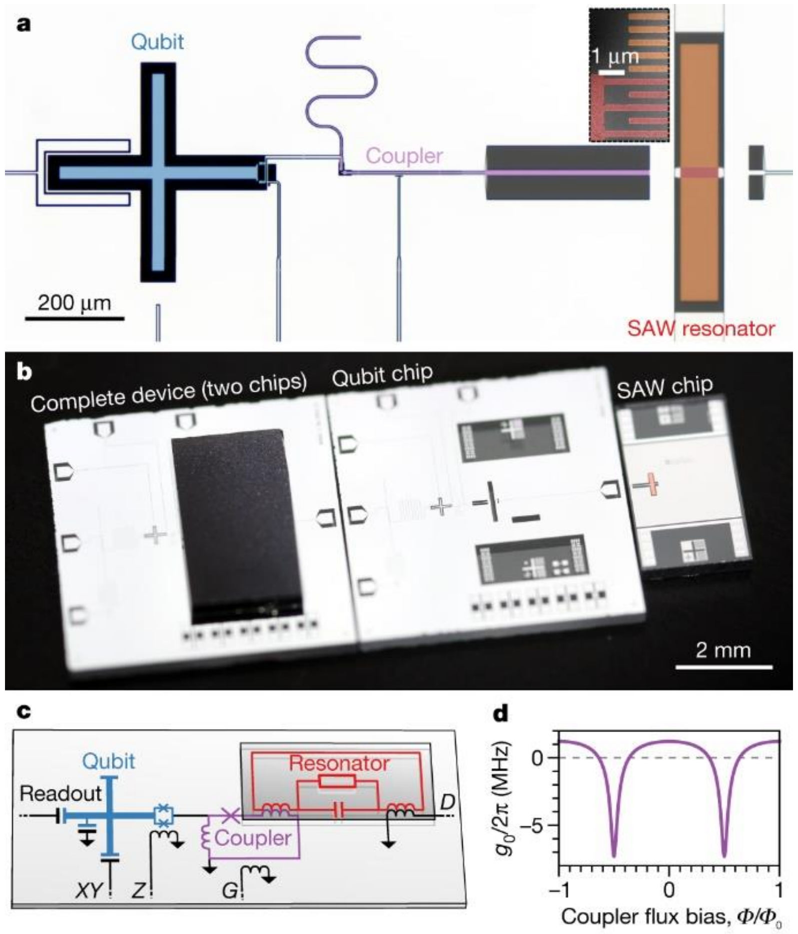
p~ 400 nm => ~5 GHz

Direct coupling to microwave quantum states !





High Frequency Acoustic Resonators Coupled to Superconducting Qubits



The **Jaynes-Cummings Hamiltonian** can also be used to describe qubit-phonon interactions

$$H_{JC} = \hbar\Omega_m \hat{a}^\dagger \hat{a} + \frac{\hbar\omega_q}{2}\sigma_z + \hbar g\left(\hat{a}^\dagger\sigma_- + \hat{a}\sigma_+\right)$$

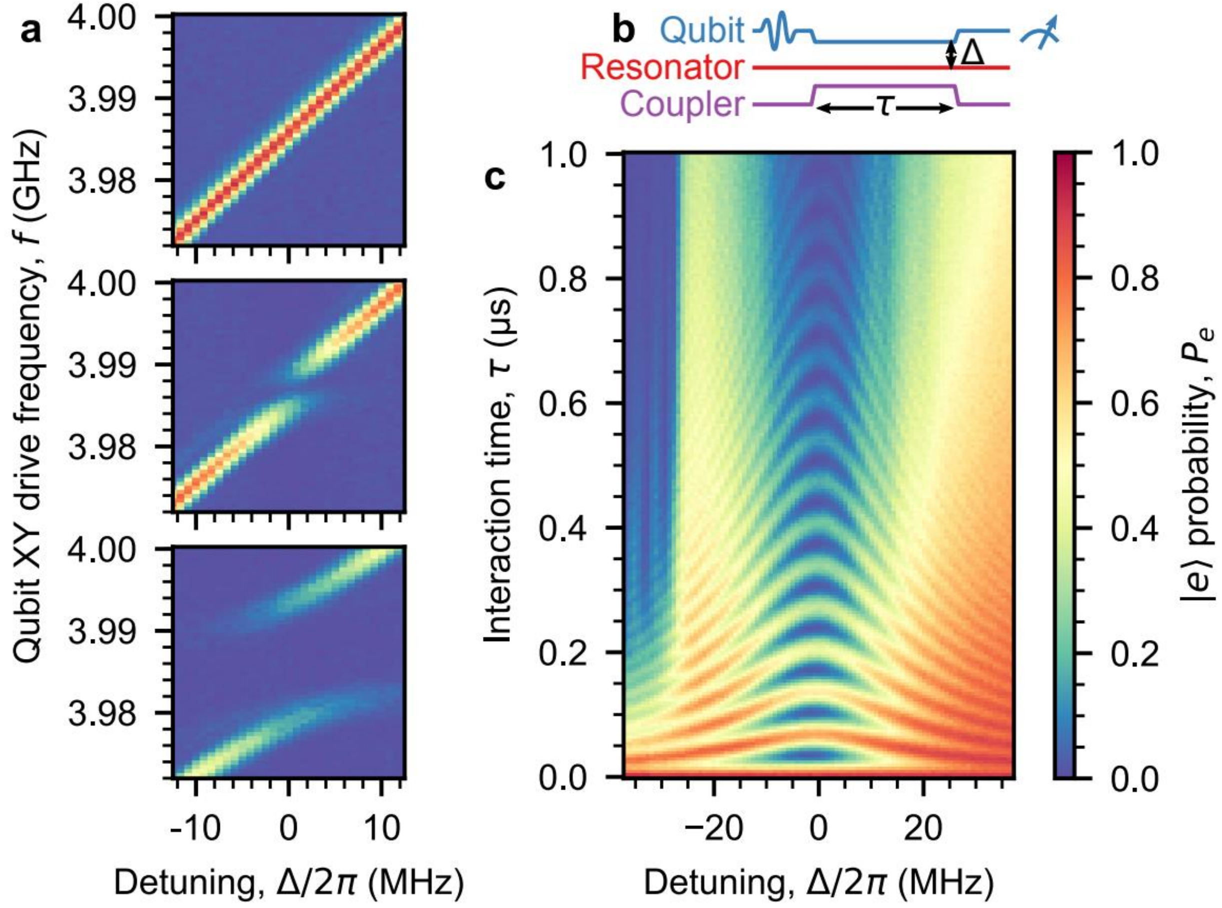
- Hamiltonian for the microwave resonator
- ω_r denotes a photon frequency

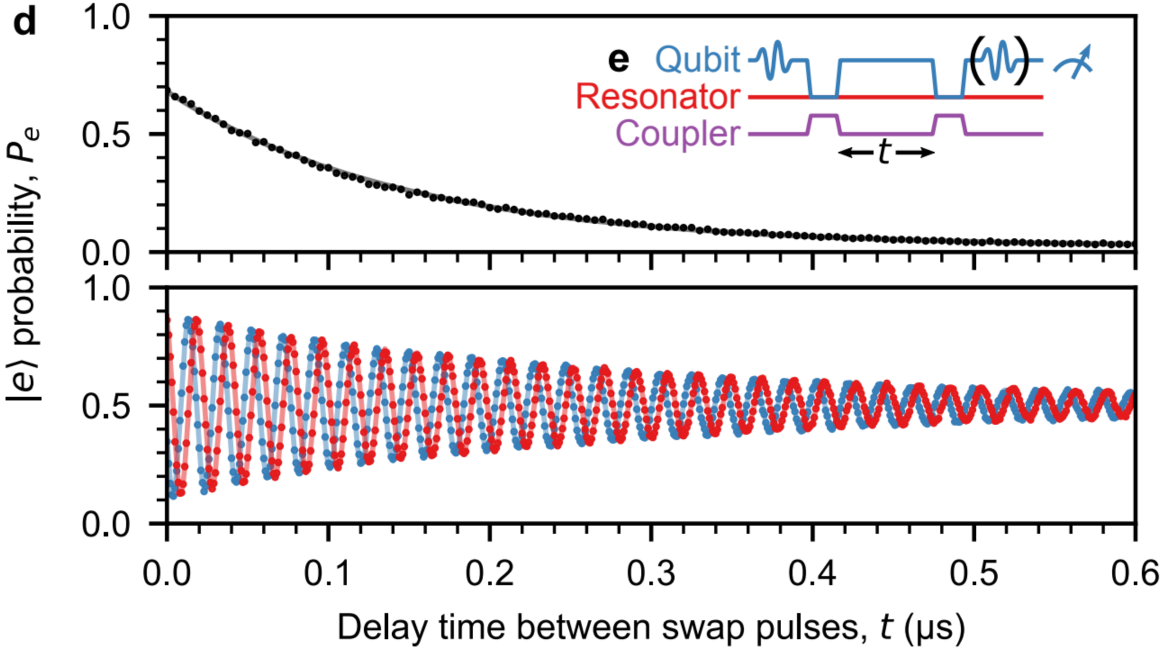
- Hamiltonian for the SC qubit when approximated as a two-level system
- ω_q denotes the qubit frequency

- qubit-phonon interaction Hamiltonian
- g denotes the qubit-photon coupling

Quantum control of surface acoustic wave phonons, *Nature* **563**, 661-665 (2018)

High Frequency Acoustic Resonators Coupled to Superconducting Qubits

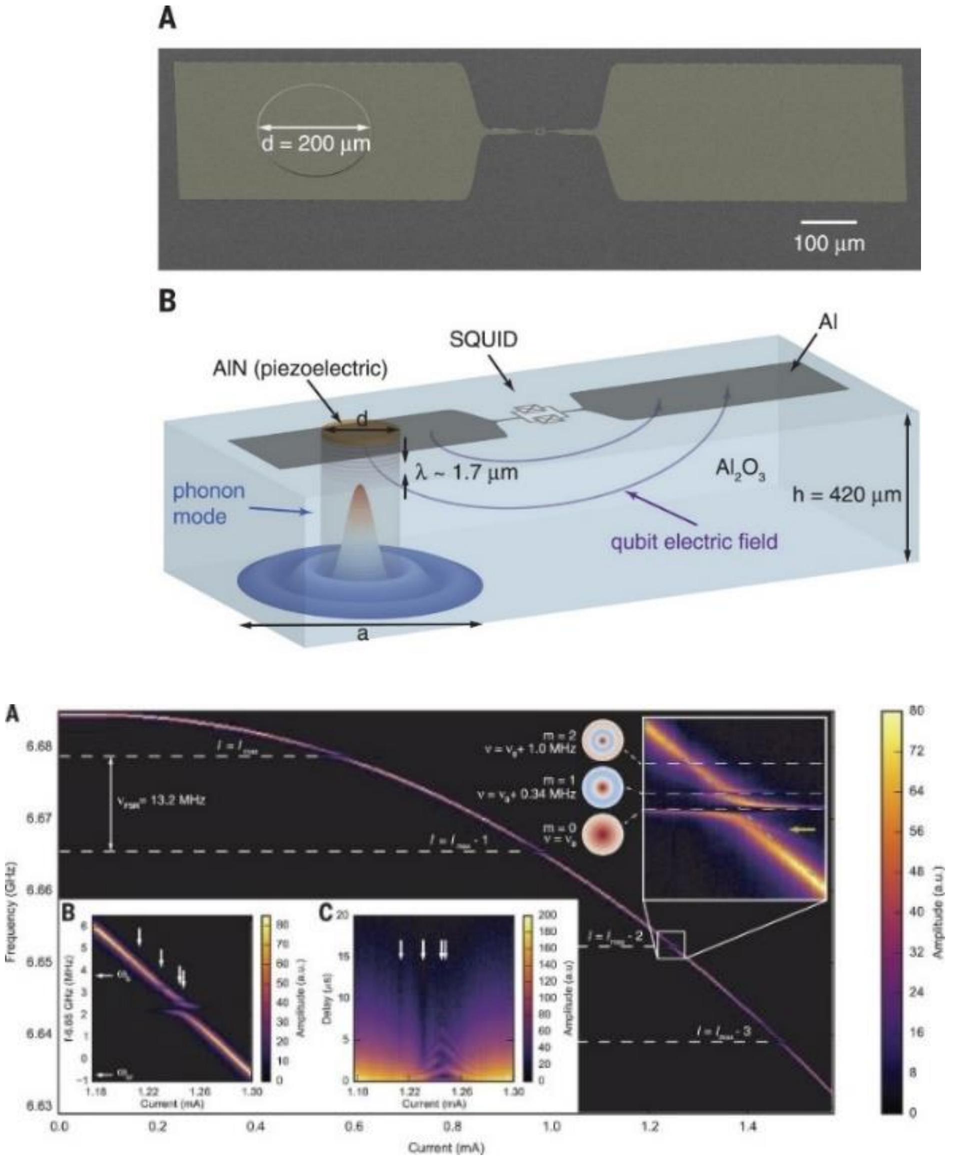




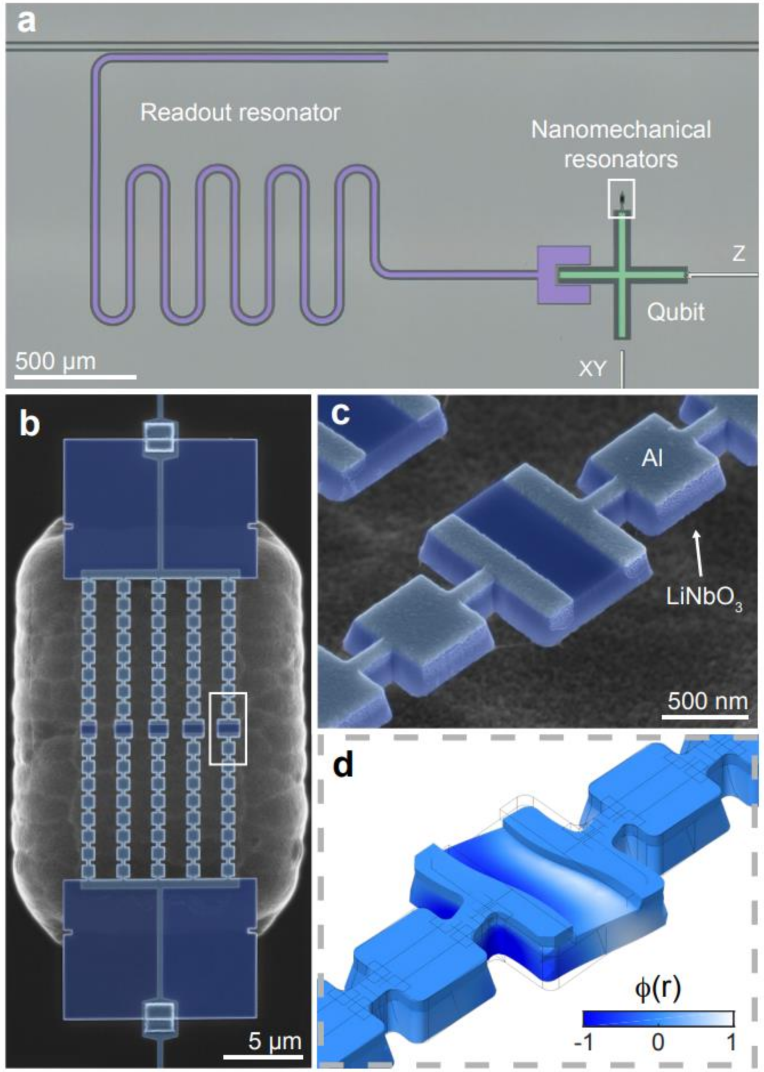
- Surface acoustic wave phonon can be coupled to superconducting qubits via piezoelectricity
- In this device, the coupling between phonon and qubit is tunable via a coupler

Quantum control of surface acoustic wave phonons, *Nature* **563**, 661-665 (2018)

High Frequency Acoustic Resonators Coupled to Superconducting Qubits

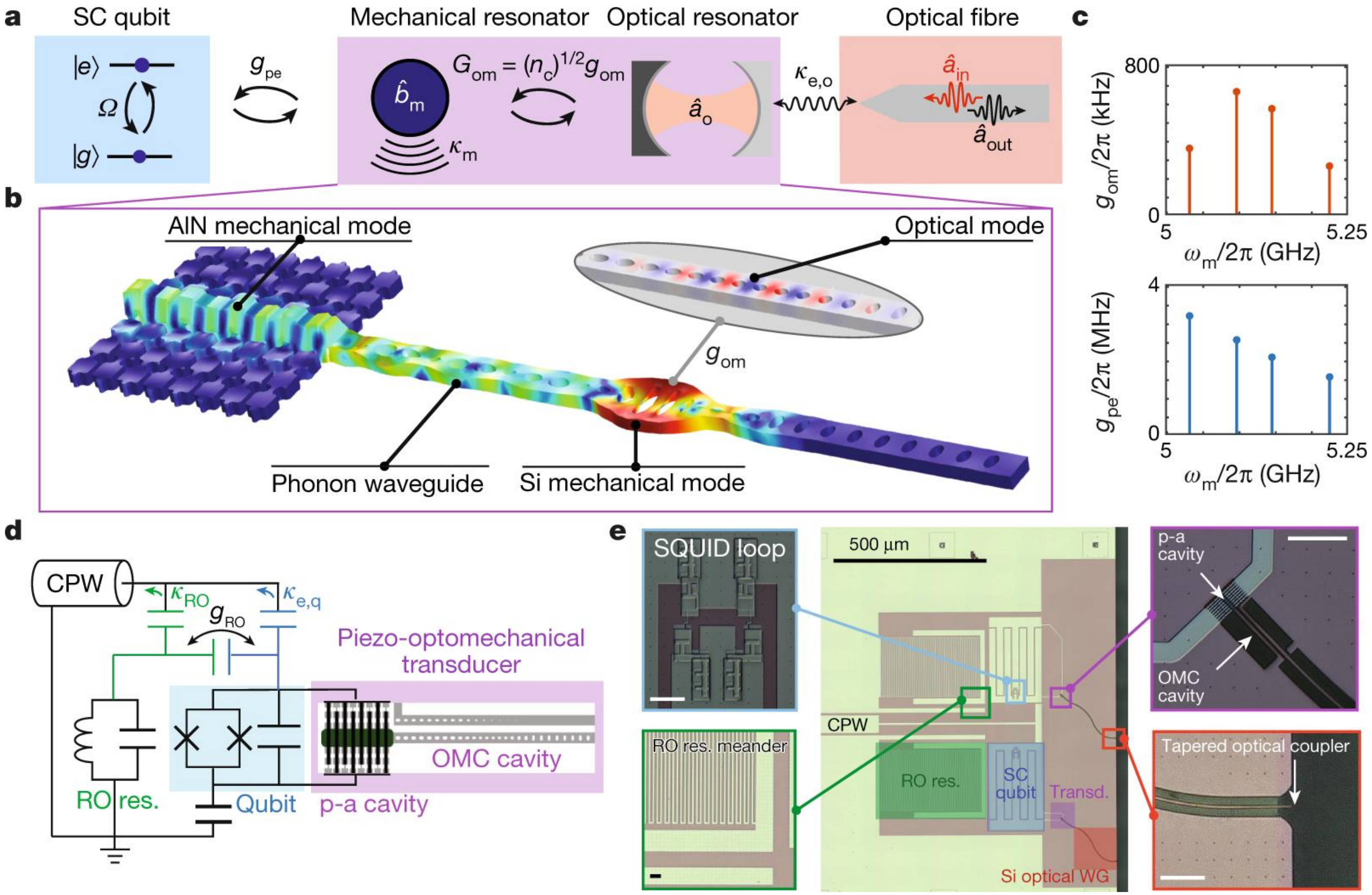


Quantum acoustics with superconducting qubits, *Science* **358**, 199-202 (2017)



Resolving the energy levels of a nanomechanical oscillator, *Nature* **571**, 537-540 (2019)

Quantum Transducer: Conversion of Microwave to Optical Signals



$$\hat{H} = \left(\frac{\hbar\omega_q}{2}\right)\sigma_z$$ **SC qubit**

$$+\ \hbar\omega_o \hat{b}^\dagger \hat{b}$$ **Optical photon**

$$+\hbar\Omega_m \hat{c}^\dagger \hat{c}$$ **Nanomechanical phonon**

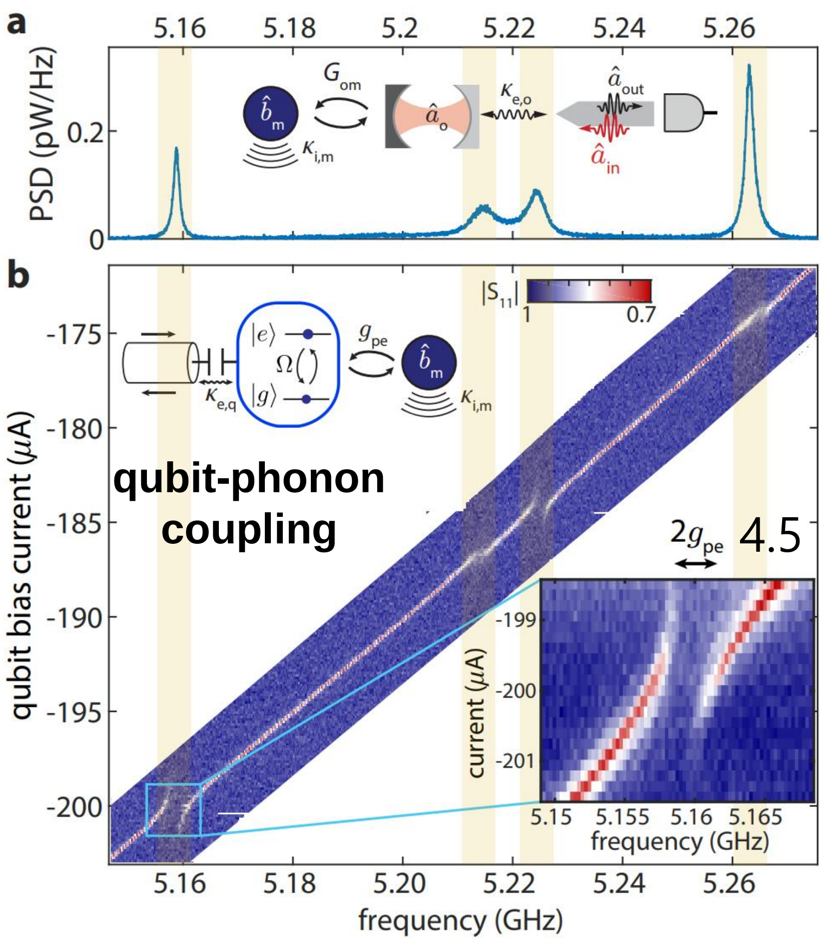
$$-\hbar g_{em}\left(\hat{c}^\dagger \sigma_- + \hat{c}\sigma_+\right)$$ **qubit-phonon coupling**

$$-\hbar g_{om}\hat{b}^\dagger \hat{b}\left(\hat{c}^\dagger + \hat{c}\right)$$ **Optomechanical coupling**

Oskar Painter group at Caltech has recently shown that a superconducting qubit can be measured using optical photons

Superconducting qubit to optical photon transduction. *Nature* **588,** 599-603 (2020)

Quantum Transducer: Conversion of Microwave to Optical Signals



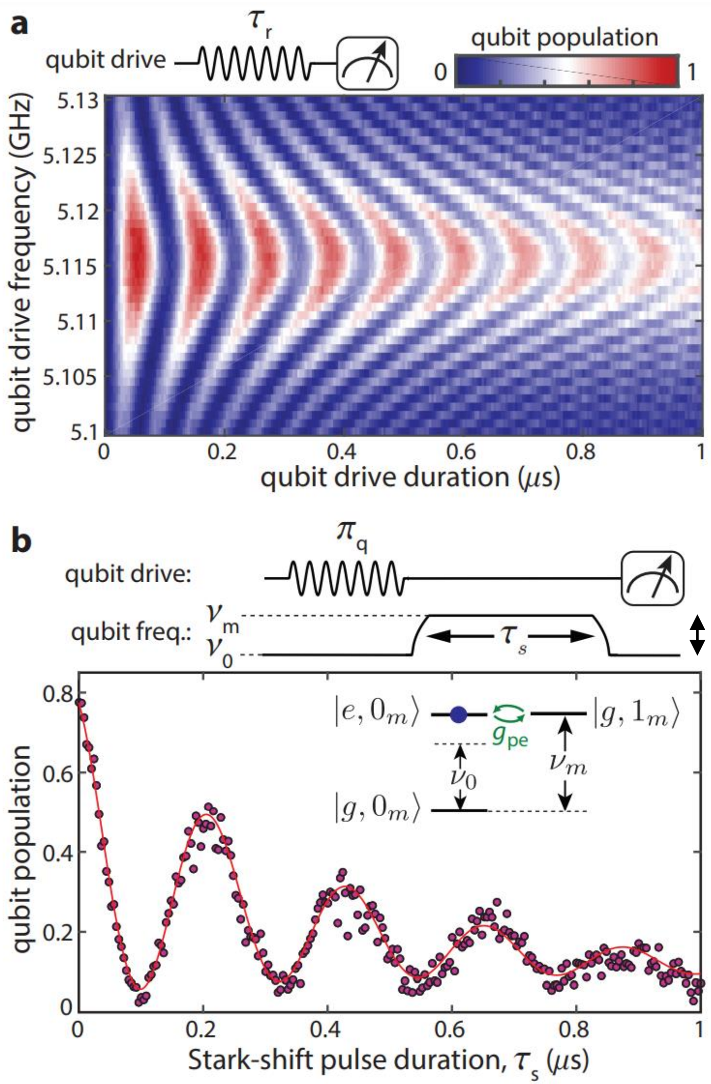
phonon sidebands from optomechanical Interaction

qubit-phonon coupling

$2g_{pe}$ 4.5 MHz

$$\frac{\omega_m}{2\pi} = (5.1588, 5.2146, 5.2242, 5.2631)\ \text{GHz}$$

$$\frac{g_{om}}{2\pi} = (420, 500, 527, 692)\ \text{GHz}$$



(dispersive) qubit readout with superconducting microwave resonators

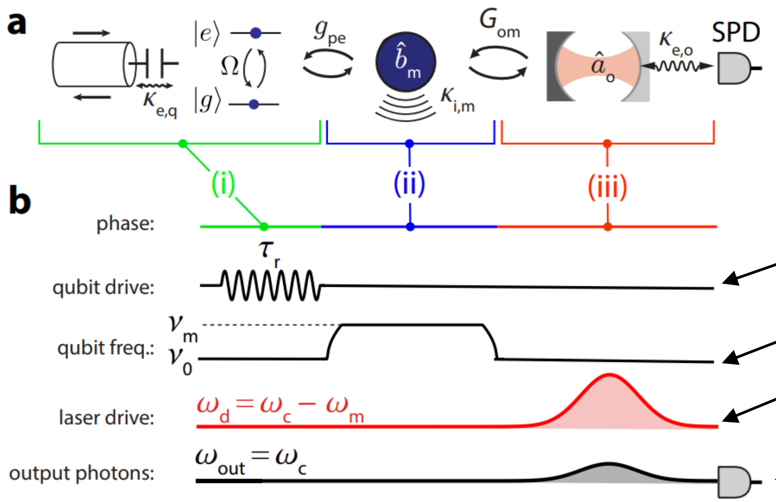
10 MHz

qubit-phonon swap

Vacuum Rabi oscillation between qubit and phonon

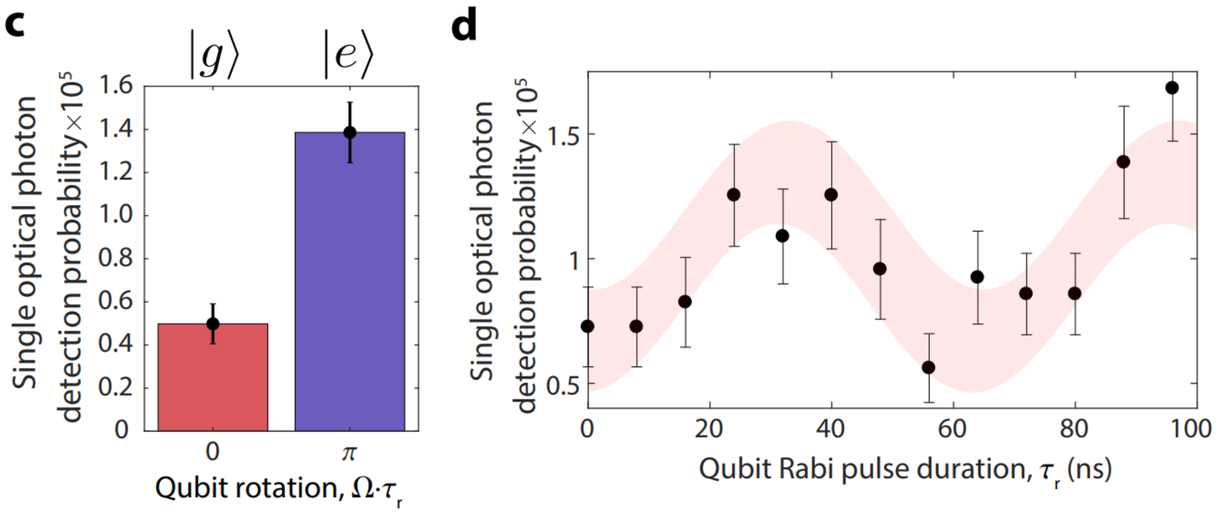
Superconducting qubit to optical photon transduction. *Nature* **588,** 599-603 (2020)

Quantum Transducer: Conversion of Microwave to Optical Signals



The pulse sequence for quantum transduction

- qubit state preparation (ex, |0> -> |1>)
- qubit-phonon swap (qubit relaxed, phonon excited)
- red-detuned optical pump to the optomechanical cavity
- sideband photon signals are measured using SNSPD.



π-pulse probability: $(1.38 \pm 0.14) \times 10^{-5}$

no π-pulse probability: $(0.5 \pm 0.09) \times 10^{-5}$

Superconducting qubit to optical photon transduction.
Nature **588,** 599-603 (2020)