Anomaly Free Condition in Elementary Particle Physics and Energy-Momentum Tensor Structure in Hadron Physics

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``What is matter made of?”

(물질이 무엇으로 이루어져 있을까?)

``What are the most fundamental particles of the Universe?”

(우주만물의 기본입자들이 무엇일까?)

``How do they interact with one another?”

(그들이 어떤 상호작용을 할까?)

The year 2015 marks the 100th anniversary of Albert Einstein's presentation of the complete Theory of General Relativity to the Prussian Academy.

A Century of General Relativity, Berlin, Germany, 30 Nov 2015 - 5 Dec 2015.

General Relativity & Gravitation: a Centennial Perspective, Pennsylvania State University, 8-12 June 2015.

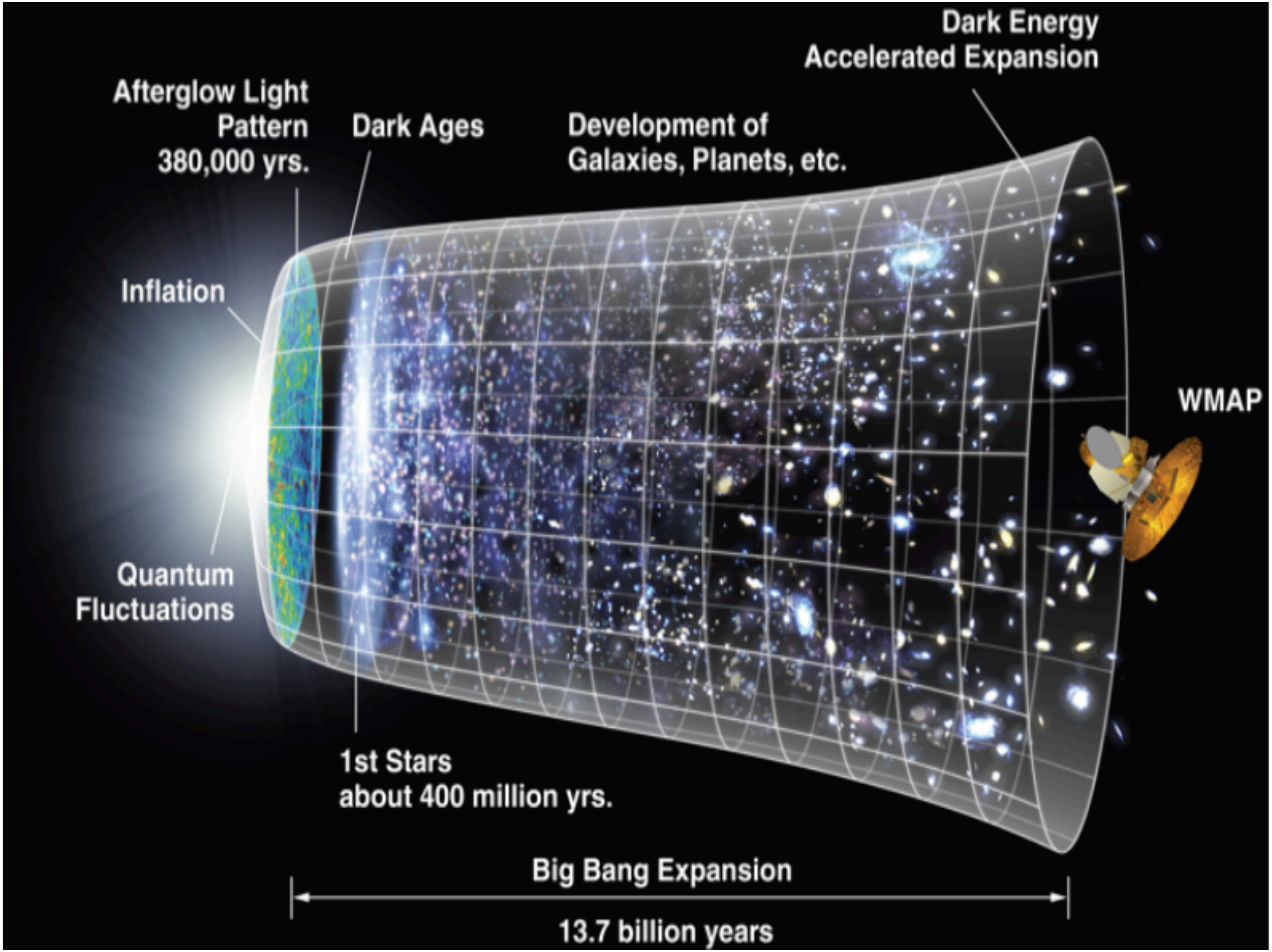
$$gt \rightarrow \frac{gt}{\sqrt{1+(gt)^2}}$$

GPS application

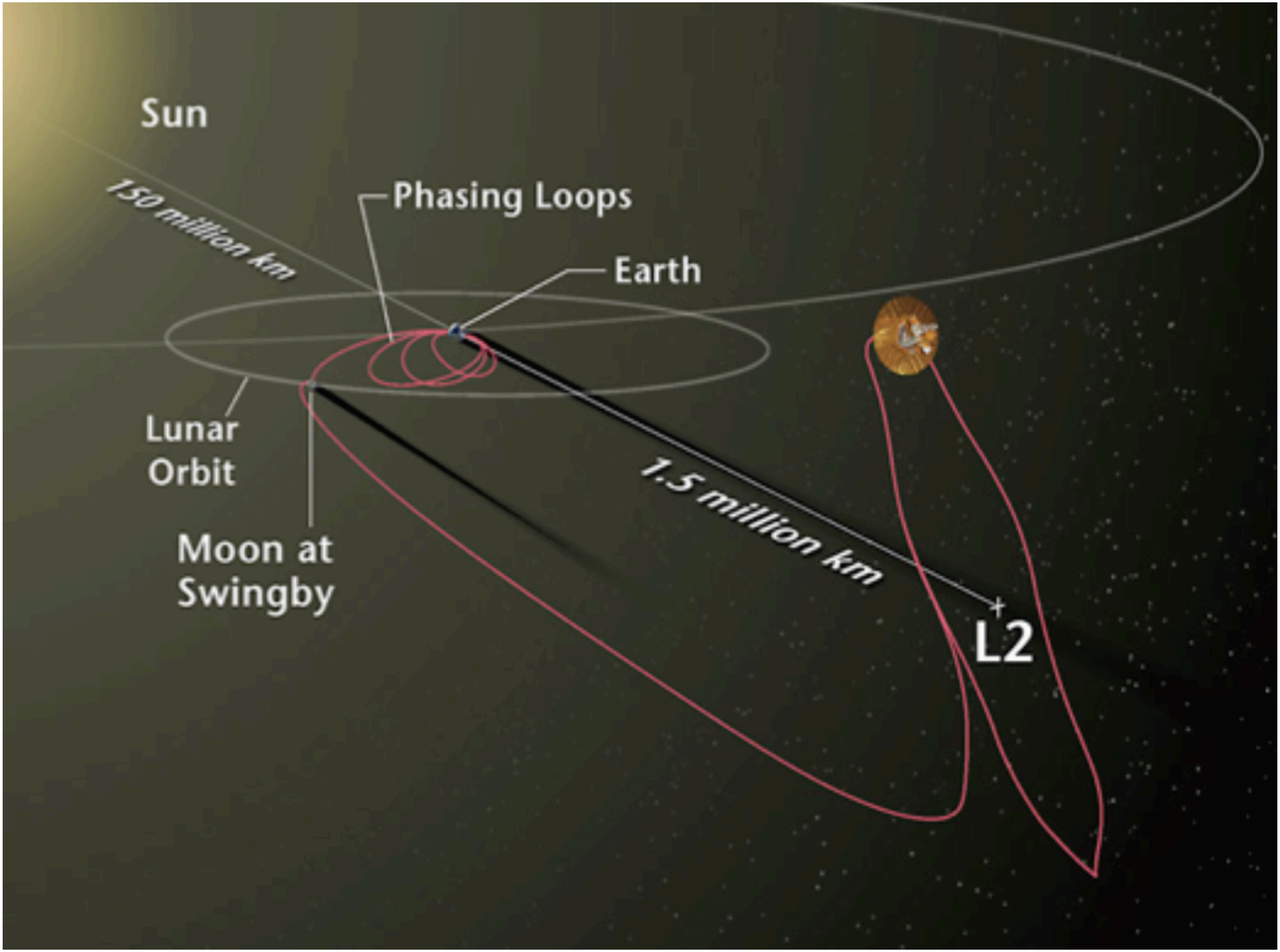
$$ds^2 = -(1+gz)^2 dt^2 + dx^2 + dy^2 + dz^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

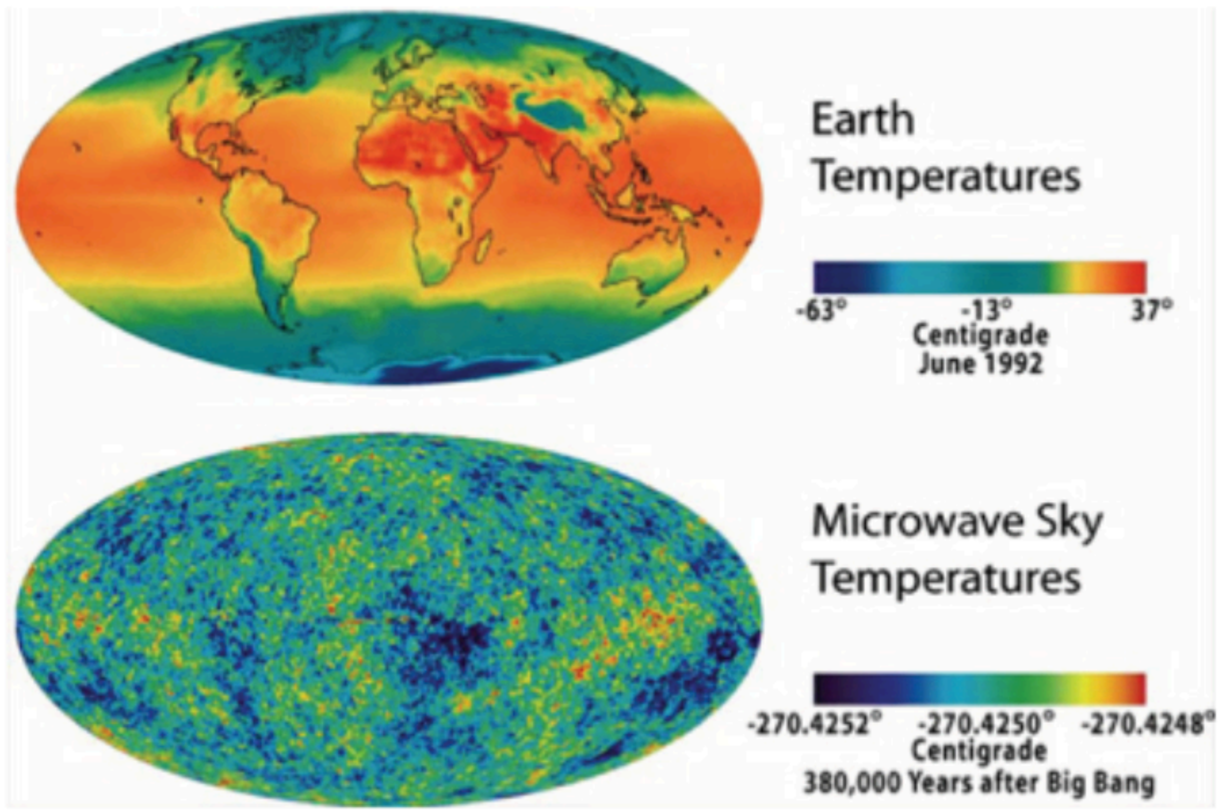
Measure of local spacetime curvature = Measure of matter energy density



Dark Energy
Accelerated Expansion
Afterglow Light
Pattern
380,000 yrs.
Dark Ages
Development of
Galaxies, Planets, etc.
Inflation
WMAP
Quantum
Fluctuations
1st Stars
about 400 million yrs.
Big Bang Expansion
13.7 billion years

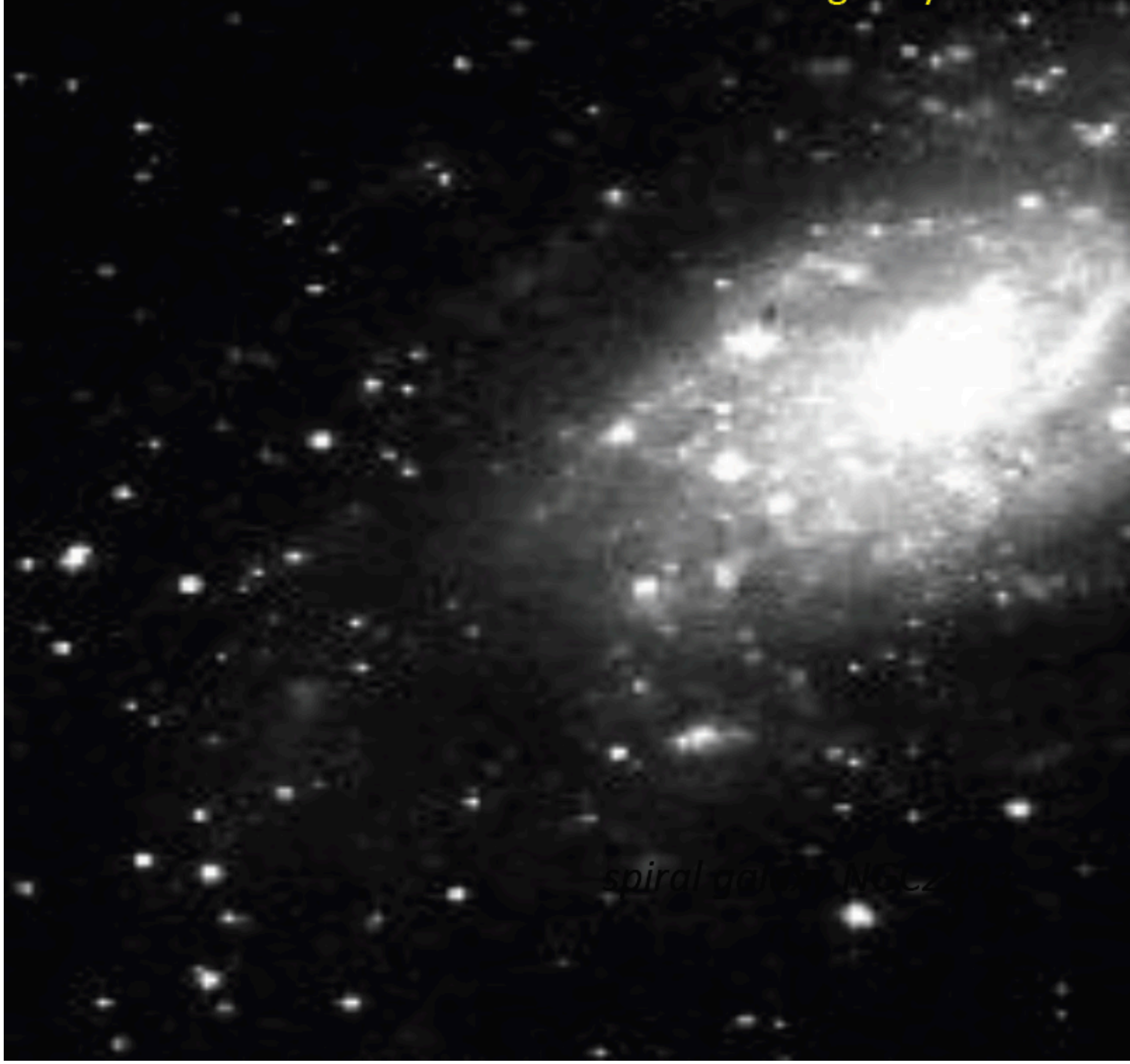


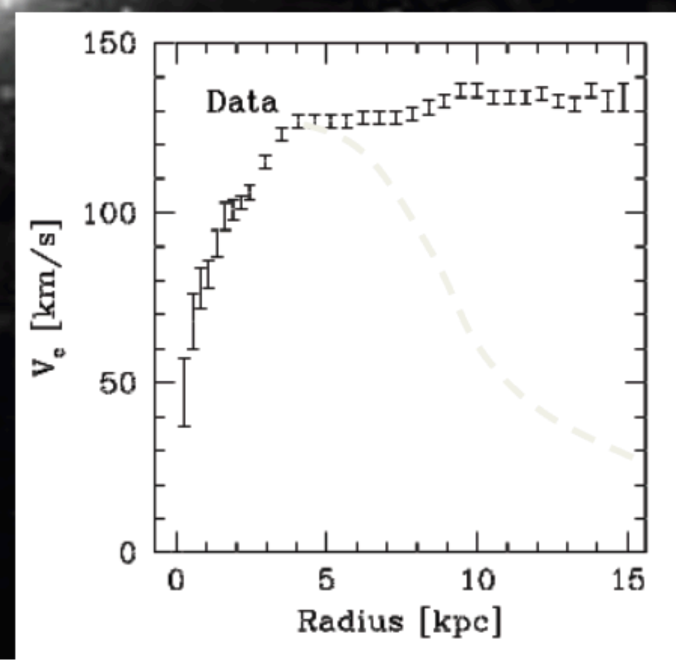
Sun
150 million km
Phasing Loops
Earth
Lunar
Orbit
Moon at
Swingby
1.5 million km
L2


Earth
Temperatures
-63°
-13°
37°
Centigrade
June 1992
Microwave Sky
Temperatures
-270.4252°
-270.4250°
-270.4248°
Centigrade
380,000 Years after Big Bang

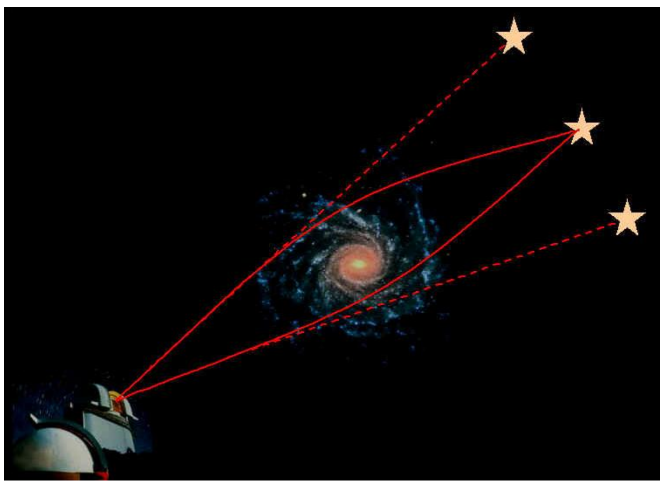
Dark Matter in Spiral Galaxies

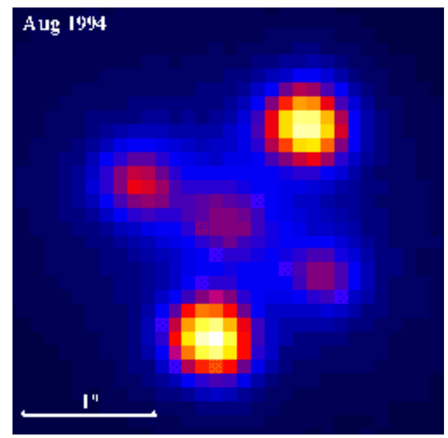
Rotation Curve represents gas/stars circular orbit velocity as function of distance from the galaxy's center

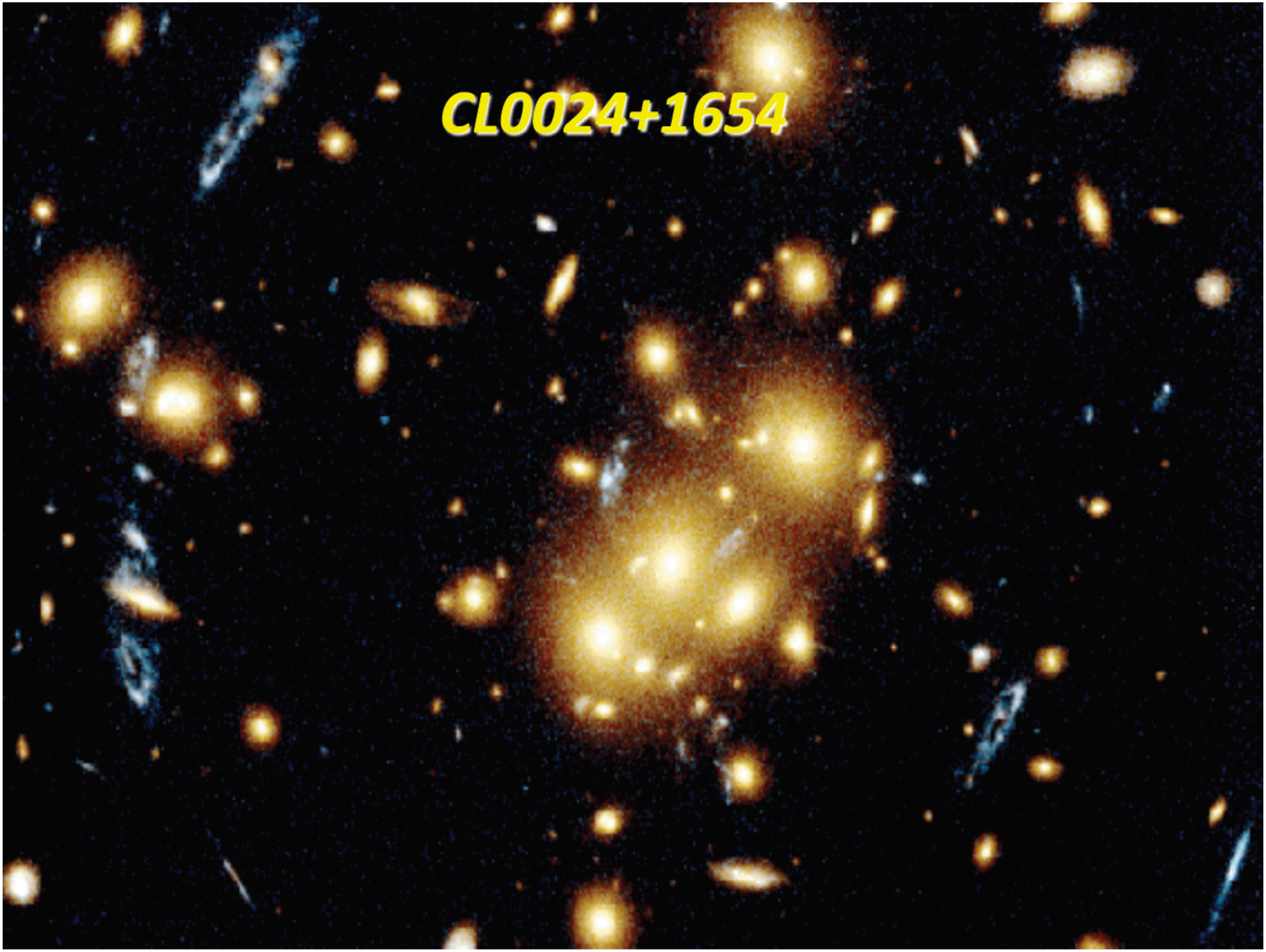




Gravitational Lensing Effect



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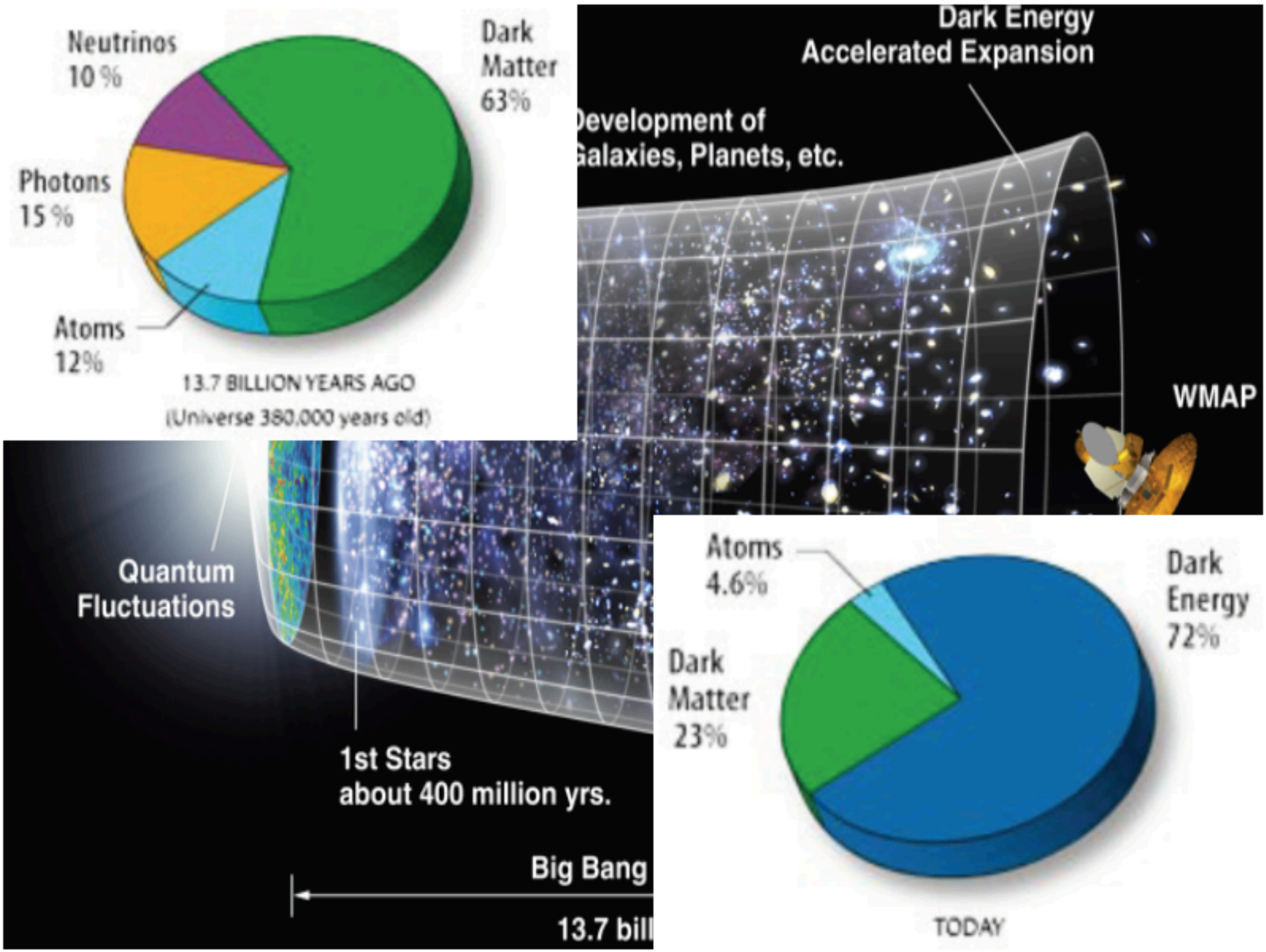
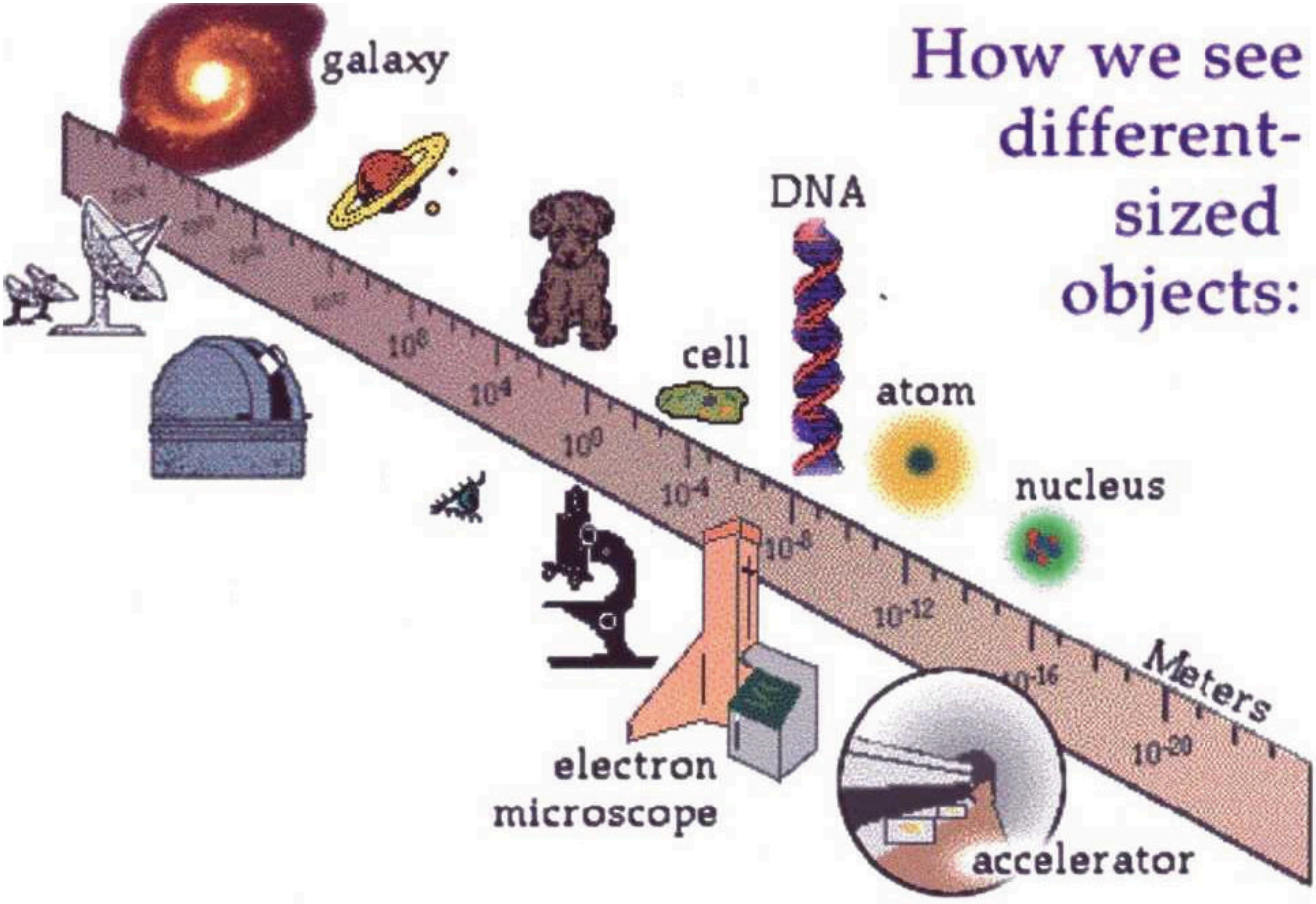


Neutrinos
10%
Dark
Matter
63%
Photons
15%
Atoms
12%
13.7 BILLION YEARS AGO
(Universe 380,000 years old)
Dark Energy
Accelerated Expansion
Development of
Galaxies, Planets, etc.
WMAP
Quantum
Fluctuations
1st Stars
about 400 million yrs.
Big Bang
13.7 bill
Atoms
4.6%
Dark
Energy
72%
Dark
Matter
23%
TODAY

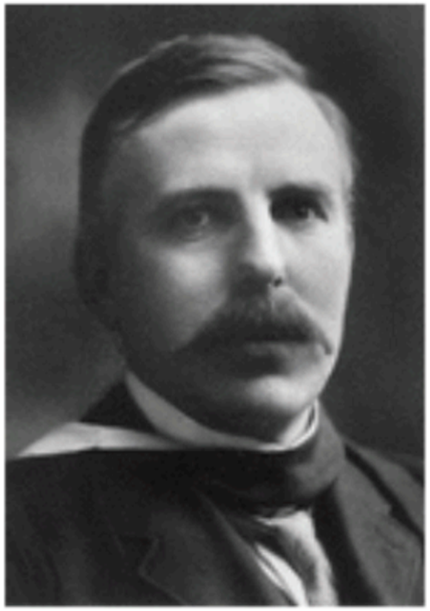
Eleven Science Questions for the 21st Century

- What is Dark Matter?
- What is the nature of Dark Energy?
- How did the Universe begin?
- Did Einstein have the last word on Gravity?
- What are the masses of the Neutrinos, and how have they shaped the evolution of the Universe?
- How do Cosmic Accelerators work and what are they accelerating?
- Are Protons unstable?
- What are the new states of matter at exceedingly High Density and Temperature?
- Are there Additional Space-Time Dimensions?
- How were the elements from Iron to Uranium made?
- Is a new theory of Matter and Light needed at the Highest Energies?

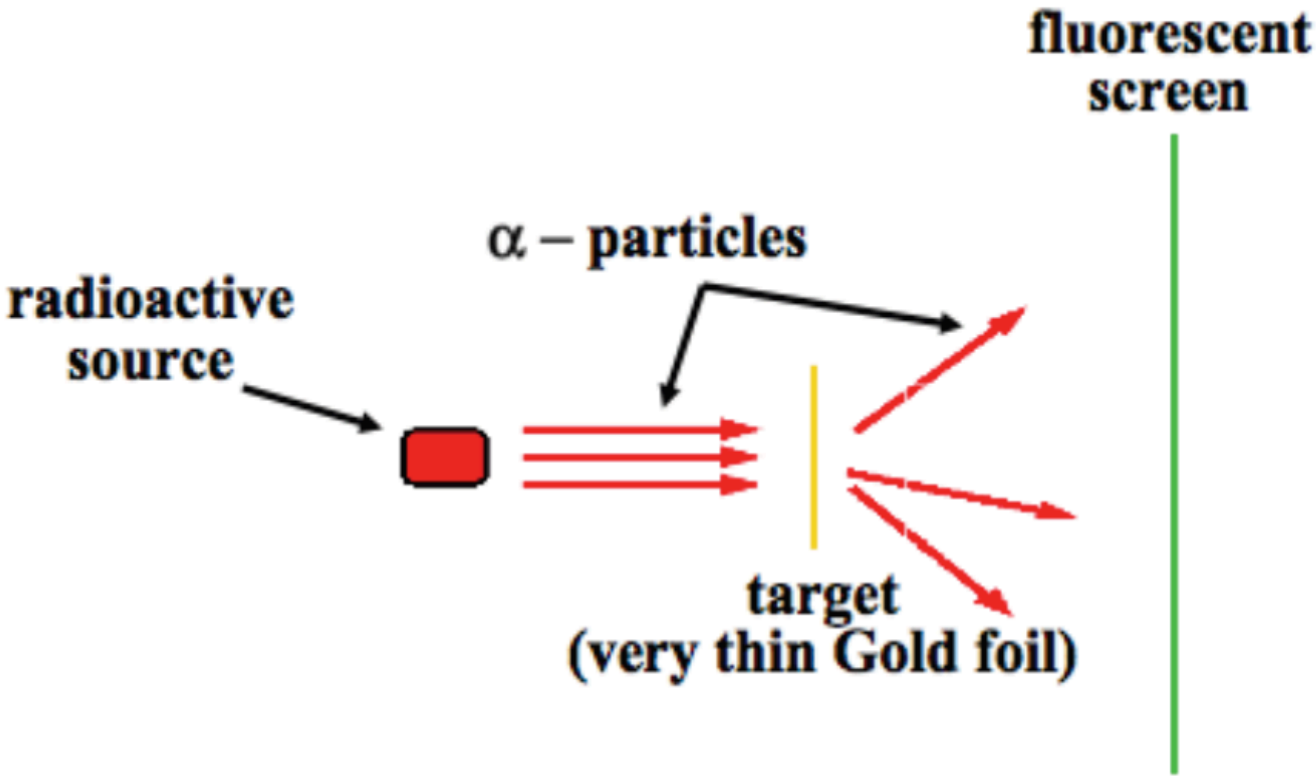


How we see different-sized objects:
galaxy
DNA
cell
atom
nucleus
electron microscope
accelerator
10^{8}
10^{4}
10^{0}
10^{-4}
10^{-8}
10^{-12}
10^{-16}
10^{-20}
Meters

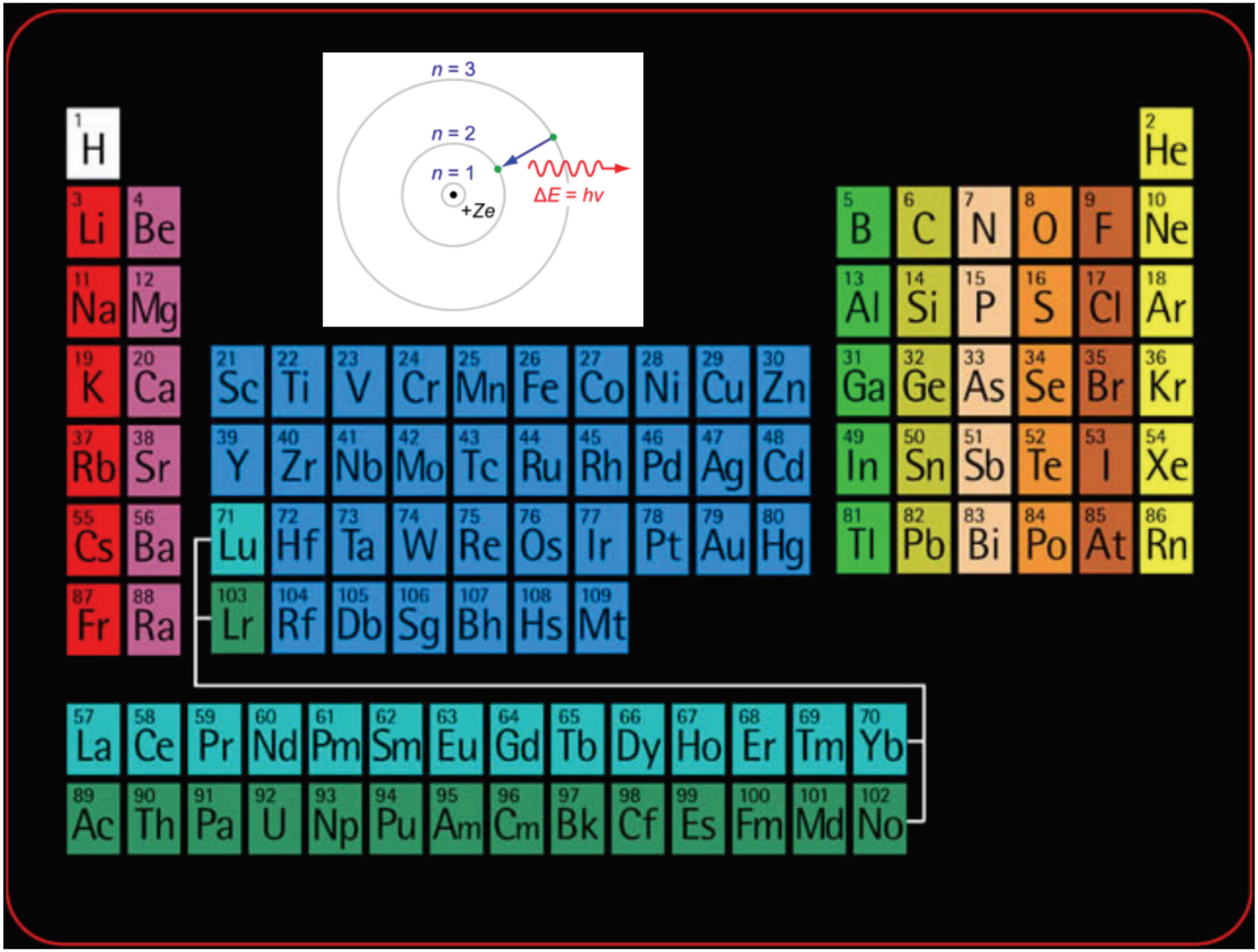
1909 – 13: Rutherford's scattering experiments
Discovery of the atomic nucleus



Ernest Rutherford



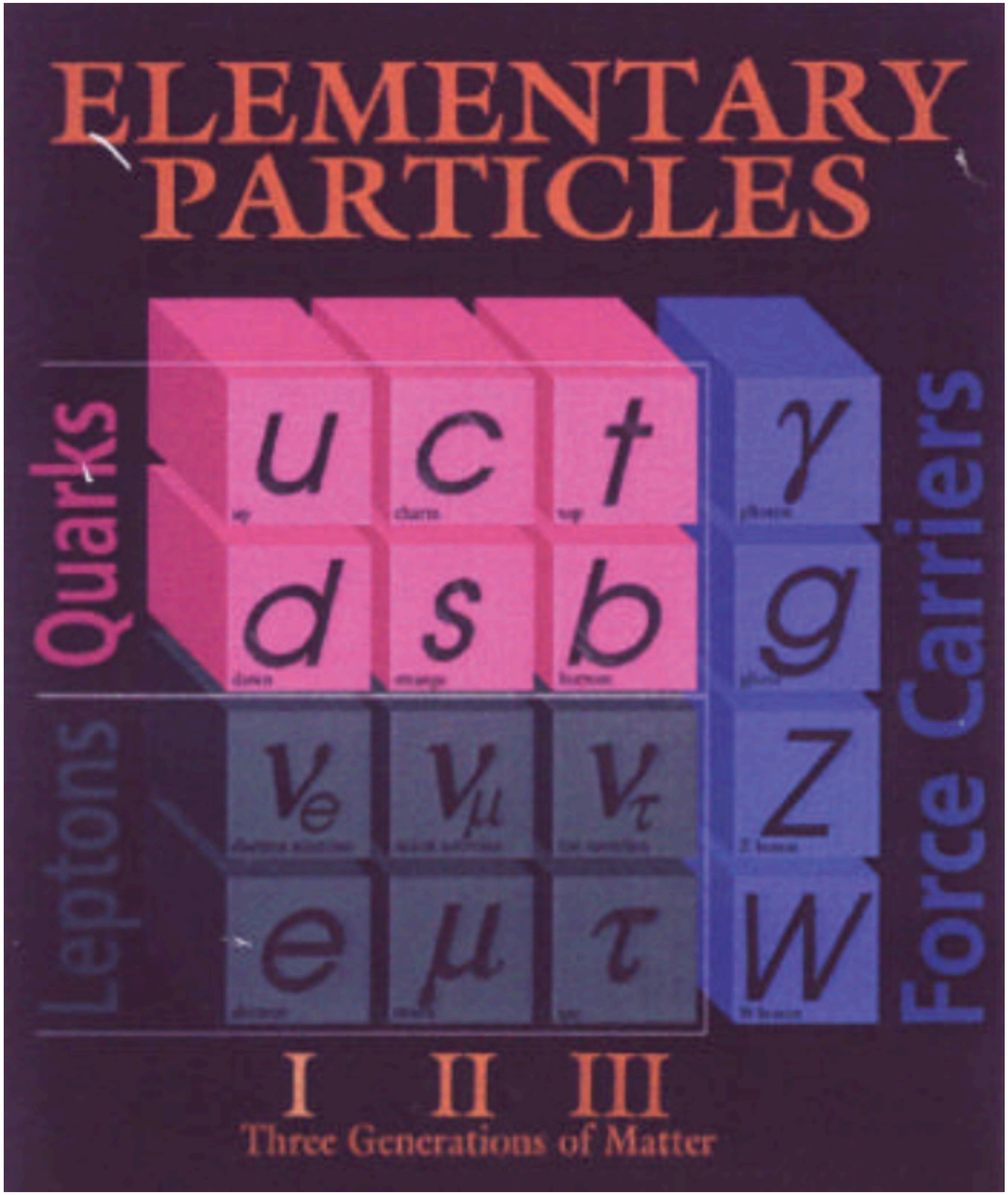
Periodic Table of the Elements



Ground-state Energy of Atom n=1 and j=1/2

$$E = m\sqrt{1-(Z\alpha)^2}$$

Dirac Vacuum cannot hold too many protons in the nucleus of atom.


ELEMENTARY
PARTICLES
Quarks
Leptons
Force Carriers
u c t γ
d s b g
ν_e ν_μ ν_τ Z
e μ τ W
I II III
Three Generations of Matter

Standard Model

$$\begin{matrix} 2/3 \\ -1/3 \\ 0 \\ -1 \end{matrix} \quad \begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix} \quad \begin{pmatrix} \nu_e \\ e \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}$$

$$\sum_f Q_f = 0 \quad (Anomaly-Free\ \ Condition)$$

Relativity provides a Bottom-Up Fitness Test of Model Theories.

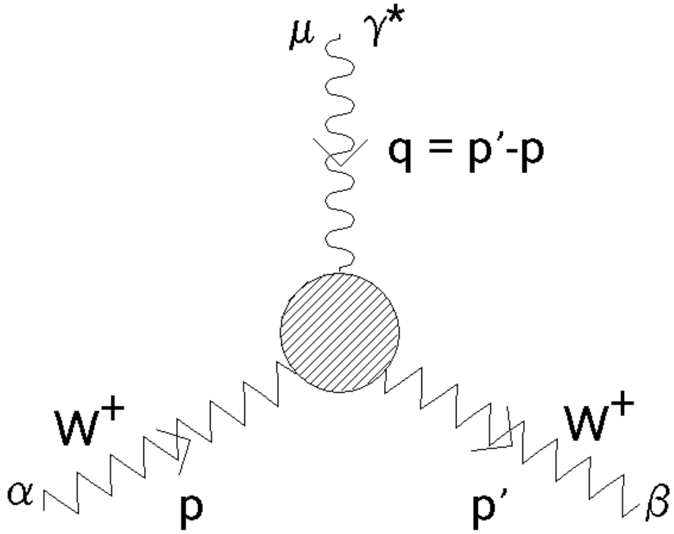
B.Bakker and C.Ji, Phys. Rev. D71,053005 (2005)

$$\left(\frac{2}{3}-\frac{1}{3}\right)\times 3+(0-1)=0$$

Three colors are necessary for the quarks: Quantum Chromodynamics

CP-Even Electromagnetic Form Factors of W± Gauge Bosons

$$\Gamma^{\mu}_{\alpha\beta} = ie\left\{A\left[(p+p')^{\mu} g_{\alpha\beta} + 2(q_{\beta} g^{\mu}_{\alpha} - q_{\alpha} g^{\mu}_{\beta})\right] + (\Delta\kappa)(g^{\mu}_{\alpha} q_{\beta} - g^{\mu}_{\beta} q_{\alpha}) + \frac{(\Delta Q)}{2M_W^2}(p+p')^{\mu} q_{\alpha} q_{\beta}\right\}$$



At tree level, for any q^2,

$$A = 1, \quad \Delta\kappa = 0, \quad \Delta Q = 0$$

Beyond tree level,

$$A = F_1(q^2),$$

$$-(\Delta\kappa) = F_2(q^2) + 2F_1(q^2),$$

$$-(\Delta Q) = F_3(q^2),$$

$$\Gamma^{\mu}_{\alpha\beta} = -ie\, J^{\mu}_{\alpha\beta}$$

$$J^{\mu}_{\alpha\beta} = \left\{-(p+p')^{\mu} g_{\alpha\beta} F_1(q^2) + (g^{\mu}_{\alpha} q_{\beta} - g^{\mu}_{\beta} q_{\alpha}) F_2(q^2) + \frac{q_{\alpha} q_{\beta}}{2M_W^2}(p+p')^{\mu} F_3(q^2)\right\}$$

Manifestly Covariant Results

$$(F_3)_{SMR} = (F_3)_{PV1} = (F_3)_{PV2} = (F_3)_{DR4}$$

$$(F_2 + 2F_1)_{SMR} = (F_2 + 2F_1)_{DR4} + \frac{g^2 Q_f}{4\pi^2}\left(\frac{1}{6}\right)$$

$$(F_2 + 2F_1)_{PV1} = (F_2 + 2F_1)_{DR4} + \frac{g^2 Q_f}{4\pi^2}\left(\frac{2}{3}\right)$$

$$(F_2 + 2F_1)_{PV2} = (F_2 + 2F_1)_{DR4} + \frac{g^2 Q_f}{4\pi^2}\left(-\frac{1}{3}\right)$$

$$g^2 = \frac{G_F M_W^2}{\sqrt{2}}$$

LFD Results for Anomaly

$$(F_2+2F_1)^{+0}_{SMR} = (F_2+2F_1)^{00}_{SMR} = (F_2+2F_1)^{\mathrm{cov}}_{SMR} = (F_2+2F_1)_{DR4} + \frac{g^2 Q_f}{4\pi^2}\left(\frac{1}{6}\right)$$

$$(F_2+2F_1)^{+0}_{PV1} = (F_2+2F_1)^{00}_{PV1} = (F_2+2F_1)^{\mathrm{cov}}_{PV1} = (F_2+2F_1)_{DR4} + \frac{g^2 Q_f}{4\pi^2}\left(\frac{2}{3}\right)$$

$$(F_2+2F_1)^{\mathrm{cov}}_{PV2} = (F_2+2F_1)_{DR4} + \frac{g^2 Q_f}{4\pi^2}\left(-\frac{1}{3}\right)$$

Energy-Momentum Tensor

- Poincaré invariance entails that the Energy-Momentum Tensor is divergence-free, *i.e*. it defines a conserved current:

$$\partial_\mu T_{\mu\nu} = 0$$

$T_{\mu\nu}$ can *always* be made symmetric

- Noether current associated with a global scale transformation:

  $x \rightarrow e^{-\sigma x}$

  is the dilation current: $D_{\mu\nu}=T_{\mu\nu}\, x_\nu$

- In a scale invariant theory, the dilation current is conserved

$$\partial_\mu \mathcal{D}_\mu = 0 = [\partial_\mu T_{\mu\nu}] x_\nu + T_{\mu\nu}\delta_{\mu\nu}$$
$$= T_{\mu\mu}\ ,$$

- Consequently, the *energy-momentum tensor is traceless* in a scale invariant theory.

Electromagnetic Field Tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

Lagrangian: $L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)$

Lorenz Force EOM Example

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & 0 & 0 \\ E_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$f^{\mu} = m\frac{du^{\mu}}{d\tau} = qF^{\mu\nu}u_{\nu} \quad \text{where} \quad u^{\mu} = \frac{dx^{\mu}}{d\tau}$$

$$\frac{d^2t(\tau)}{d\tau^2} = \frac{qE_x}{m}\frac{dx(\tau)}{d\tau}$$

$$\frac{d^2x(\tau)}{d\tau^2} = \frac{qE_x}{m}\frac{dt(\tau)}{d\tau}$$

$$\frac{d^2y(\tau)}{d\tau^2} = 0$$

$$\frac{d^2z(\tau)}{d\tau^2} = 0$$

$$t(\tau) = \frac{m}{qE_x}u_t \sinh\left(\frac{qE_x\tau}{m}\right)$$

$$x(\tau) = \frac{m}{qE_x}u_t\left\{-1 + \cosh\left(\frac{qE_x\tau}{m}\right)\right\}$$

$$y(\tau) = 0$$

$$z(\tau) = u_z\tau$$

Parabola →Hyperbola (Relativity: u.u=1 or v < c)

Newtonian gravity→Einstein's GR: $gt \to \frac{gt}{\sqrt{1+(gt)^2}}$

Coupled EOMs

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & 0 & 0 \\ E_x & 0 & 0 & B_y \\ 0 & 0 & 0 & 0 \\ 0 & -B_y & 0 & 0 \end{pmatrix}$$

$$\frac{d^2t(\tau)}{d\tau^2} = \frac{qE_x}{m}\frac{dx(\tau)}{d\tau}$$
$$\frac{d^2x(\tau)}{d\tau^2} = \frac{q}{m}\left(E_x\frac{dt(\tau)}{d\tau} - B_y\frac{dz(\tau)}{d\tau}\right)$$
$$\frac{d^2y(\tau)}{d\tau^2} = 0$$
$$\frac{d^2z(\tau)}{d\tau^2} = \frac{qB_y}{m}\frac{dx(\tau)}{d\tau}$$

Spacetime Interpolation

$$x^{\hat{\mu}} = \mathscr{R}^{\hat{\mu}}{}_{\nu} x^{\nu}$$

$$\mathscr{R}^{\hat{\mu}}{}_{\nu} = \begin{pmatrix} \cos\delta & 0 & 0 & \sin\delta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\delta & 0 & 0 & -\cos\delta \end{pmatrix}$$

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \longrightarrow g^{\hat{\mu}\hat{\nu}} = \begin{bmatrix} \cos 2\delta & 0 & 0 & \sin 2\delta \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin 2\delta & 0 & 0 & -\cos 2\delta \end{bmatrix}$$

$$F^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 0 & -E_x\cos\delta - B_y\sin\delta & -E_y\cos\delta + B_x\sin\delta & -E_z \\ E_x\cos\delta + B_y\sin\delta & 0 & -B_z & B_y\cos\delta - E_x\sin\delta \\ E_y\cos\delta - B_x\sin\delta & B_z & 0 & -B_x\cos\delta - E_y\sin\delta \\ E_z & -B_y\cos\delta + E_x\sin\delta & B_x\cos\delta + E_y\sin\delta & 0 \end{pmatrix}$$

Resolution of coupled EOMs with interpolation

$$F^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 0 & -E_x\cos\delta - B_y\sin\delta & 0 & 0 \\ E_x\cos\delta + B_y\sin\delta & 0 & 0 & -B_y\cos\delta + E_x\sin\delta \\ 0 & 0 & 0 & 0 \\ 0 & B_y\cos\delta - E_x\sin\delta & 0 & 0 \end{pmatrix}$$

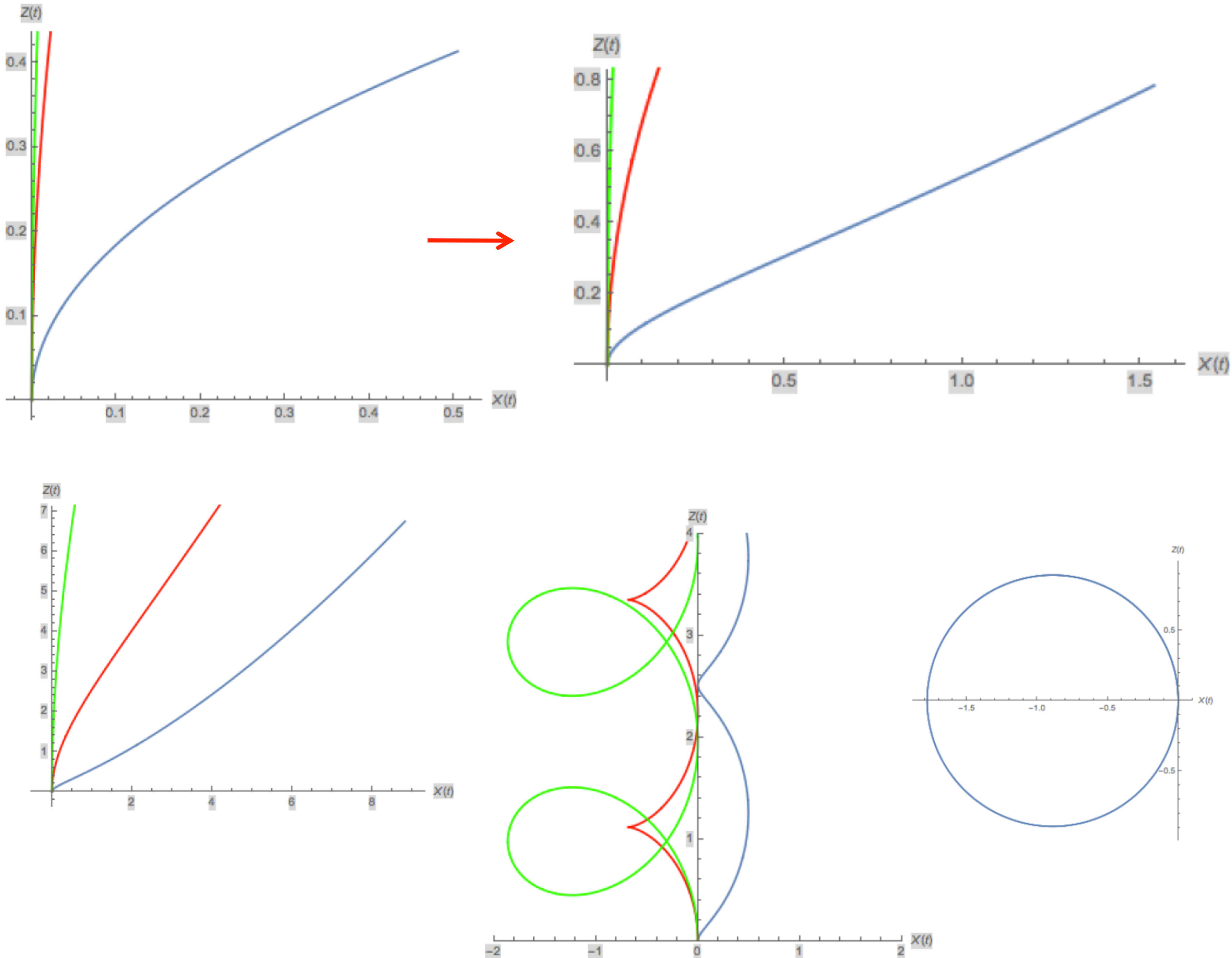
$$E_x\sin\delta - B_y\cos\delta = 0$$

$$t(\tau) = \frac{1}{q(E_x^2 - B_y^2)^{3/2}}\left\{qB_y\sqrt{E_x^2 - B_y^2}(E_x u_z^0 - B_y u_t^0)\tau + mE_x(E_x u_t^0 - B_y u_z^0)\sinh\left(\frac{q\sqrt{E_x^2 - B_y^2}\tau}{m}\right)\right\}$$

$$x(\tau) = \frac{m(u_t^0 E_x - u_z^0 B_y)}{q(E_x^2 - B_y^2)}\left\{-1 + \cosh\left(\frac{q\sqrt{E_x^2 - B_y^2}\tau}{m}\right)\right\}$$

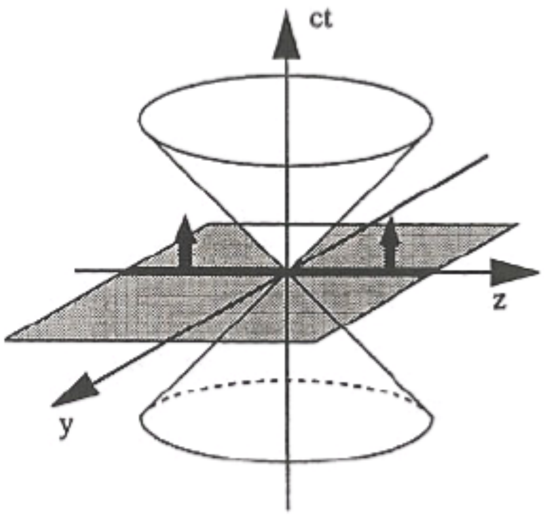
$$y(\tau) = 0$$

$$z(\tau) = \frac{1}{q(E_x^2 - B_y^2)^{3/2}}\left\{qE_x\sqrt{E_x^2 - B_y^2}(E_x u_z^0 - B_y u_t^0)\tau + mB_y(E_x u_t^0 - B_y u_z^0)\sinh\left(\frac{q\sqrt{E_x^2 - B_y^2}\tau}{m}\right)\right\}$$



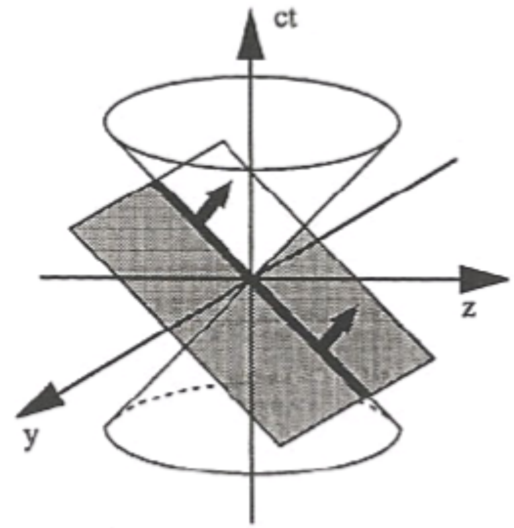
Z(t)
0.4
0.3
0.2
0.1
0.1
0.2
0.3
0.4
0.5
X(t)
Z(t)
0.8
0.6
0.4
0.2
0.5
1.0
1.5
X(t)
Z(t)
7
6
5
4
3
2
1
2
4
6
8
X(t)
Z(t)
4
3
2
1
−2
−1
0
1
2
X(t)
Z(t)
0.5
−0.5
−1.5
−1.0
−0.5
X(t)

Dirac's Proposition

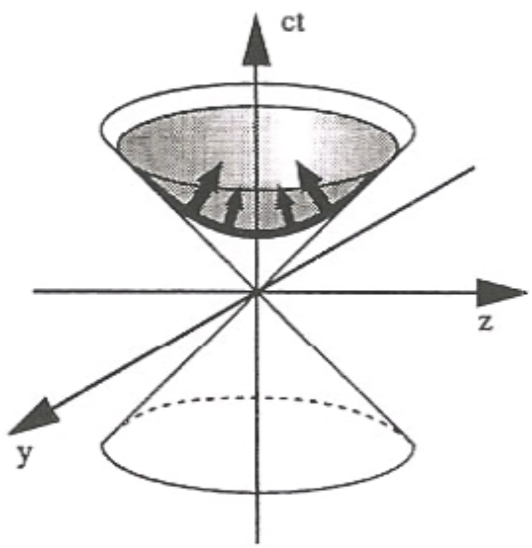


The instant form

1949

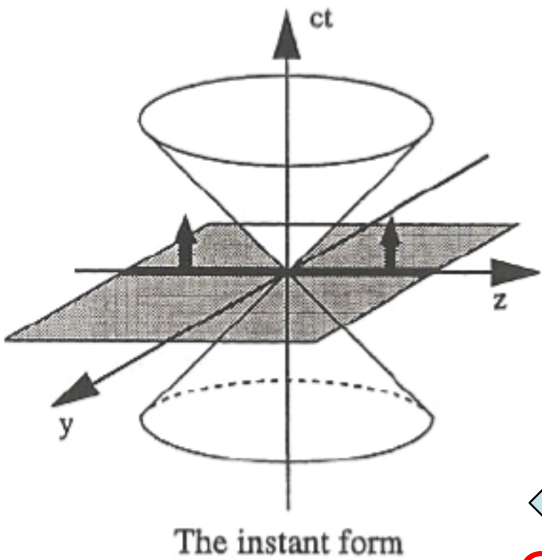


The front form

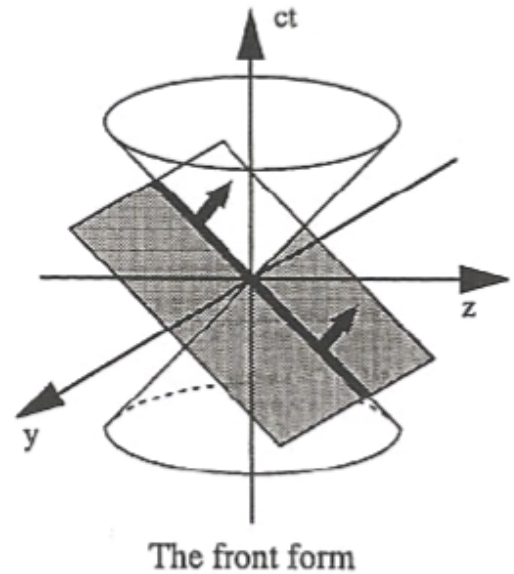


The point form

Dirac’s Proposition



1949



Can they be linked?

Traditional approach
evolved from NR dynamics

Innovative approach
for relativistic dynamics

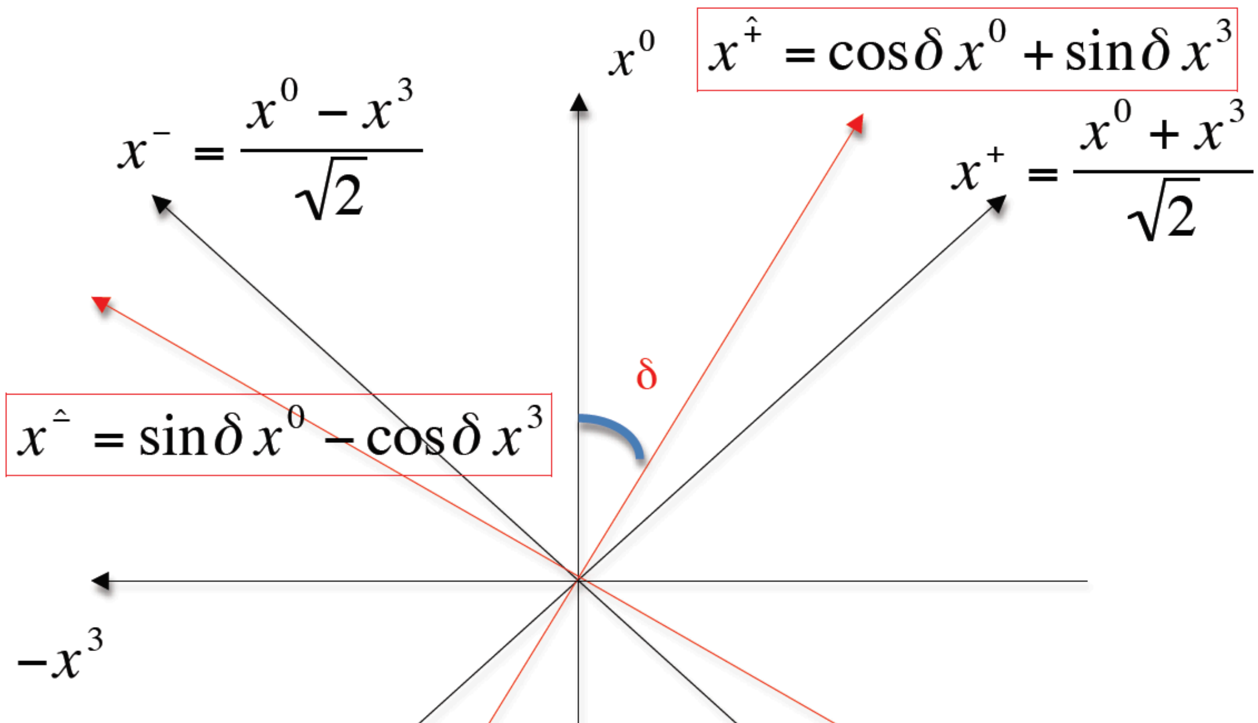
Close contact with
Euclidean space

Strictly in Minkowski space

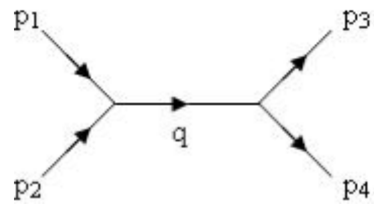
T-dept QFT, LQCD,etc.

DIS, PDFs, DVCS, GPDs, etc.

Interpolation between Instant and Front Forms



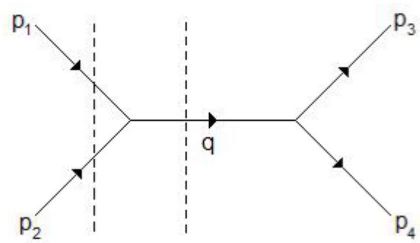
K. Hornbostel, PRD45, 3781 (1992) – RQFT
C.Ji and S.Rey, PRD53,5815(1996) – Chiral Anomaly
C.Ji and C. Mitchell, PRD64,085013 (2001) – Poincare Algebra
C.Ji and A. Suzuki, PRD87,065015 (2013) – Scattering Amps
C.Ji, Z. Li and A. Suzuki, PRD91, 065020 (2015) – EM Gauges
Z.Li, M. An and C.Ji, PRD92, 105014 (2015) – Spinors
C.Ji, Z.Li, B.Ma and A.Suzuki, in prepartion – Fermion Prop.



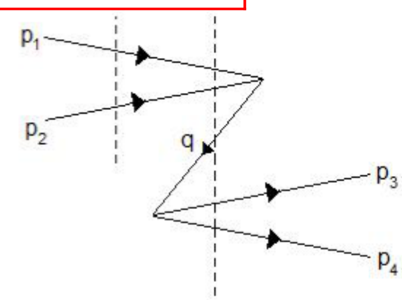
$$\delta = 0$$
$$p_0 = p^0$$
$$-p_3 = p^3$$

$$0 < \delta < \pi/4$$
$$p_{\hat{+}} = p^0 \cos\delta - p^3 \sin\delta$$
$$p_{\hat{-}} = p^0 \sin\delta + p^3 \cos\delta$$

$$\delta = \pi/4$$
$$p_+ = p^-$$
$$p_- = p^+$$



(a)



(b)

$$\frac{1}{2q^0}\left(\frac{1}{p_1^0 + p_2^0 - q^0} - \frac{1}{p_1^0 + p_2^0 + q^0}\right)$$

$$\frac{1}{2\omega_q}\left(\frac{1}{P_{\hat{+}} + \frac{\mathbb{S}q_{\hat{-}} - \omega_q}{\mathbb{C}}} - \frac{1}{P_{\hat{+}} + \frac{\mathbb{S}q_{\hat{-}} + \omega_q}{\mathbb{C}}}\right)$$

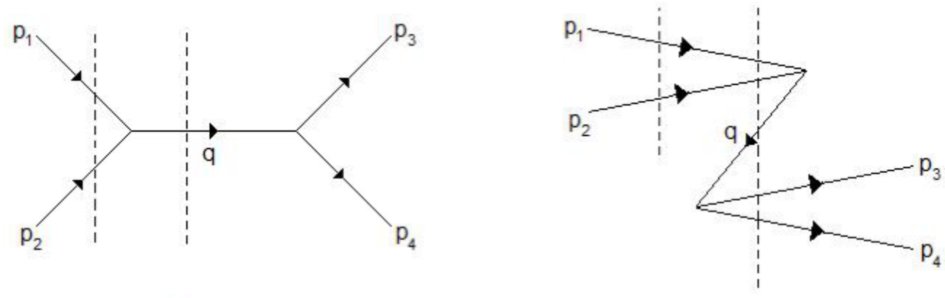
$$\omega_q = \sqrt{q_{\hat{-}}^2 + \mathbb{C}\left(\vec{\mathbf{q}}_{\hat{\perp}}^2 + m^2\right)}$$

$$\mathbb{C} = \cos 2\delta$$

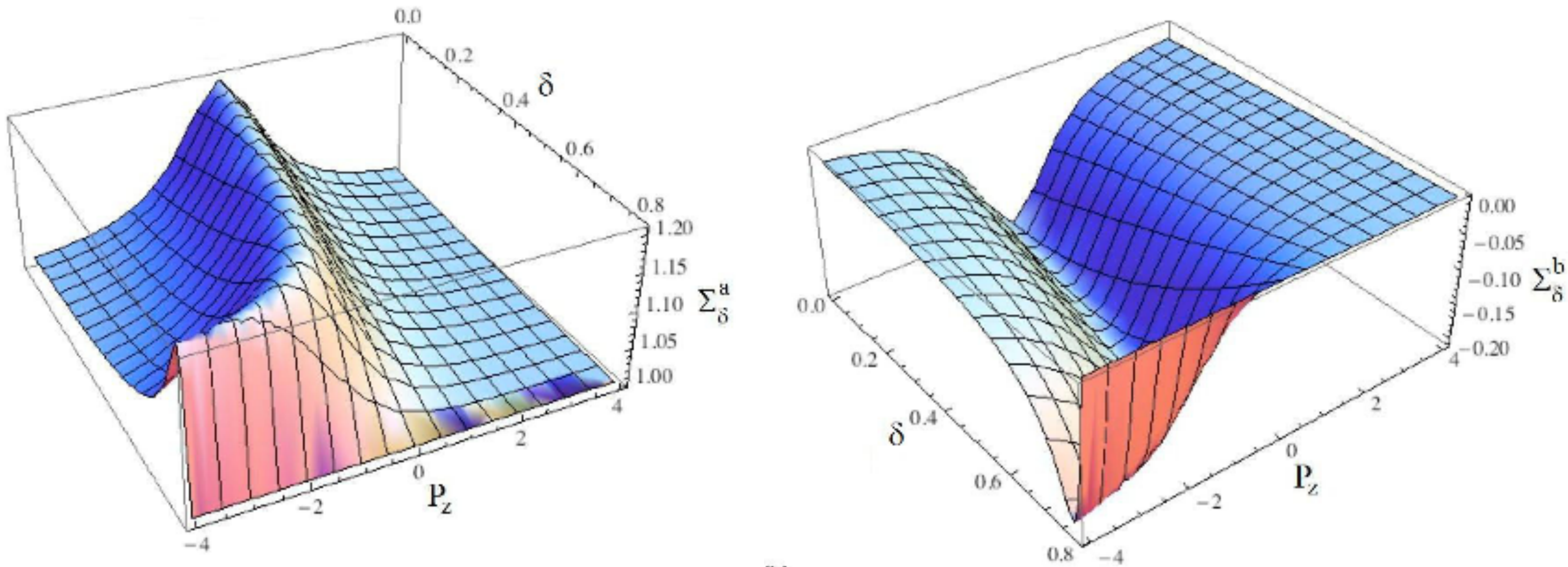
$$\mathbb{S} = \sin 2\delta$$

$$\frac{1}{P^+}\frac{1}{\left\{P^- - \frac{(\vec{\mathbf{P}}_{\perp}^2 + m^2)}{2P^+}\right\}}$$

$$\frac{\mathbb{S}q_{\hat{-}} + \omega_q}{\mathbb{C}} \to \frac{2}{\mathbb{C}} - \frac{\vec{\mathbf{q}}_{\hat{\perp}}^2 + m^2}{2q_{\hat{-}}} + \mathcal{O}(\mathbb{C})$$
$$\to \infty \quad \text{as } \mathbb{C} \to 0$$



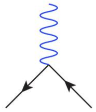
(a) (b)



(a) (b)

$\Sigma(a)+\Sigma(b)=1/(s-m^2)$; s=2 GeV, m=1GeV

J-shape peak & valley : $P_z = -\sqrt{\dfrac{s(1-C)}{2C}}$; $C = \cos(2\delta)$



Tree Level

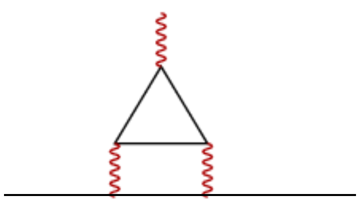
$$\partial_\mu J^\mu = 0\,;\; q_\mu \bar{u}(p')\gamma^\mu u(p)$$
$$= \bar{u}(p')[\not{p}' - \not{p}]u(p)$$
$$= \bar{u}(p')[m - m]u(p)$$
$$= 0$$

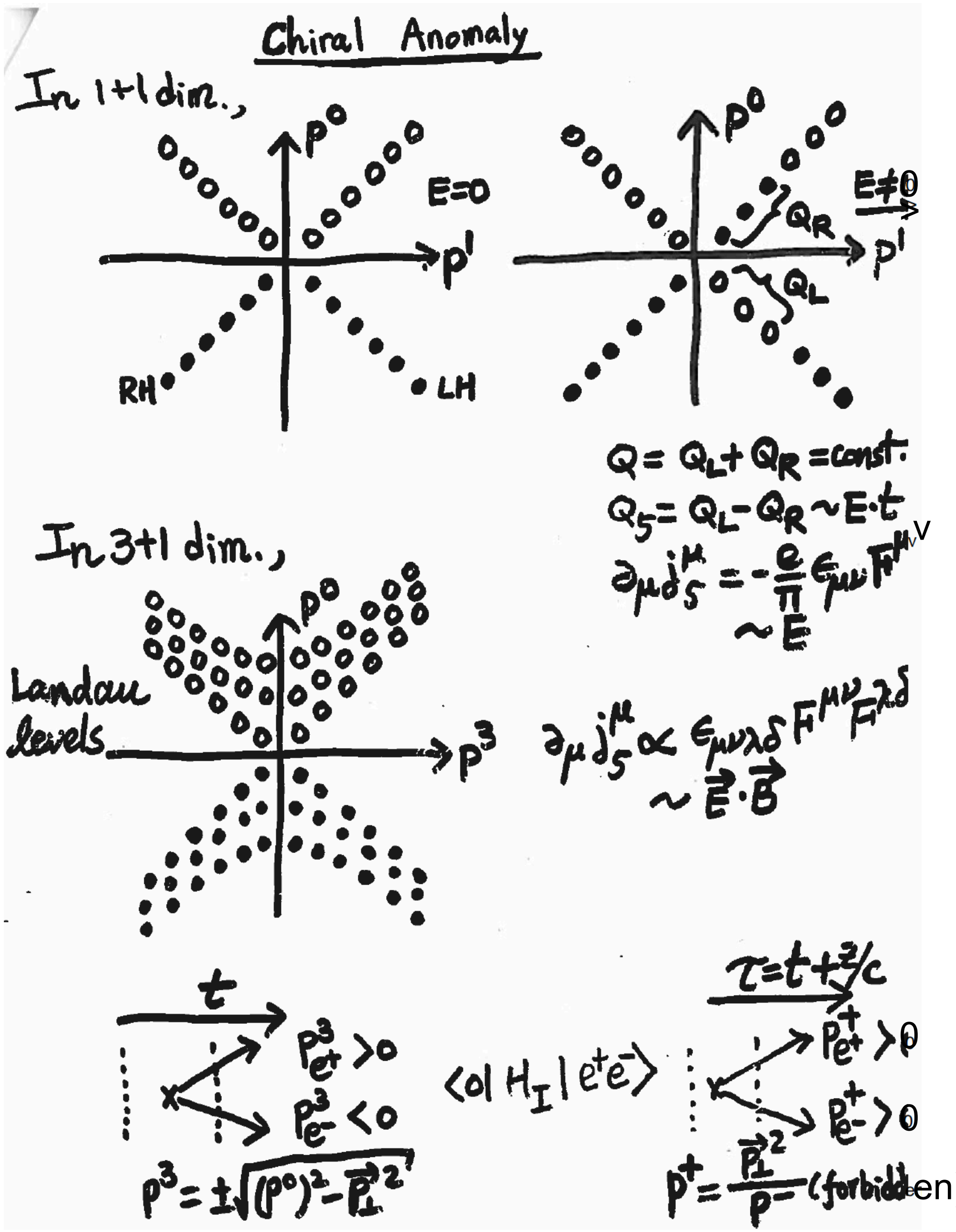
$$\partial_\mu J_5^\mu = 0\,;\; q_\mu \bar{u}(p')\gamma^\mu \gamma_5 u(p)$$
$$= \bar{u}(p')[\not{p}' - \not{p}]\gamma_5 u(p)$$
$$= \bar{u}(p')[\not{p}'\gamma_5 + \gamma_5\not{p}]u(p)$$
$$= 2m\bar{u}(p')\gamma_5 u(p)$$
$$= 0 \quad if \quad m = 0$$

Loop Level

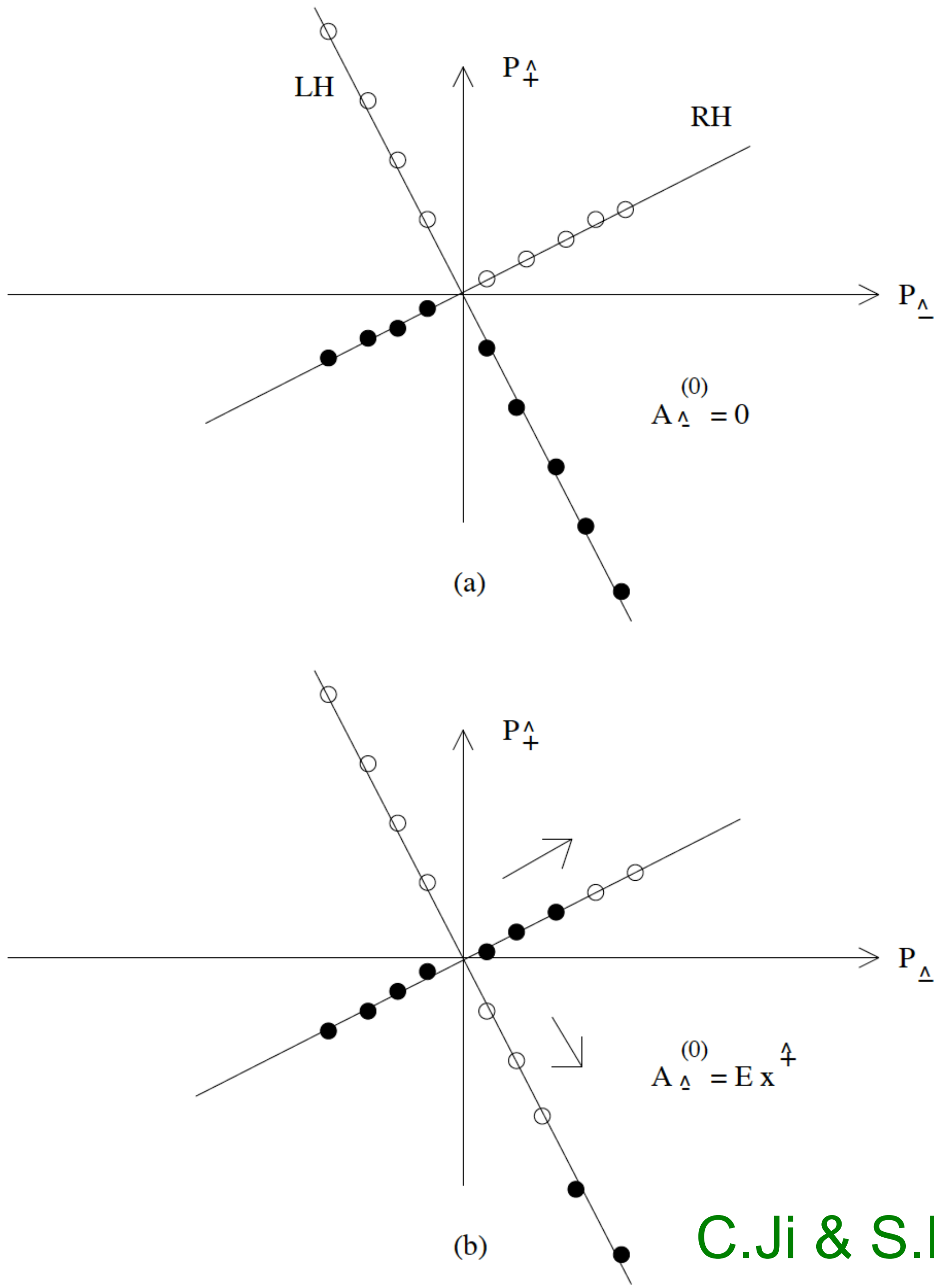
$$\partial_\mu J^\mu = 0$$

$$\partial_\mu J_5^\mu = \frac{e^2}{16\pi^2}\varepsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$$





Classical symmetry is broken due to infinite degrees of freedom in quantum fields.



C.Ji & S.Rey, PRD53,5815(1996)

Concluding Remark

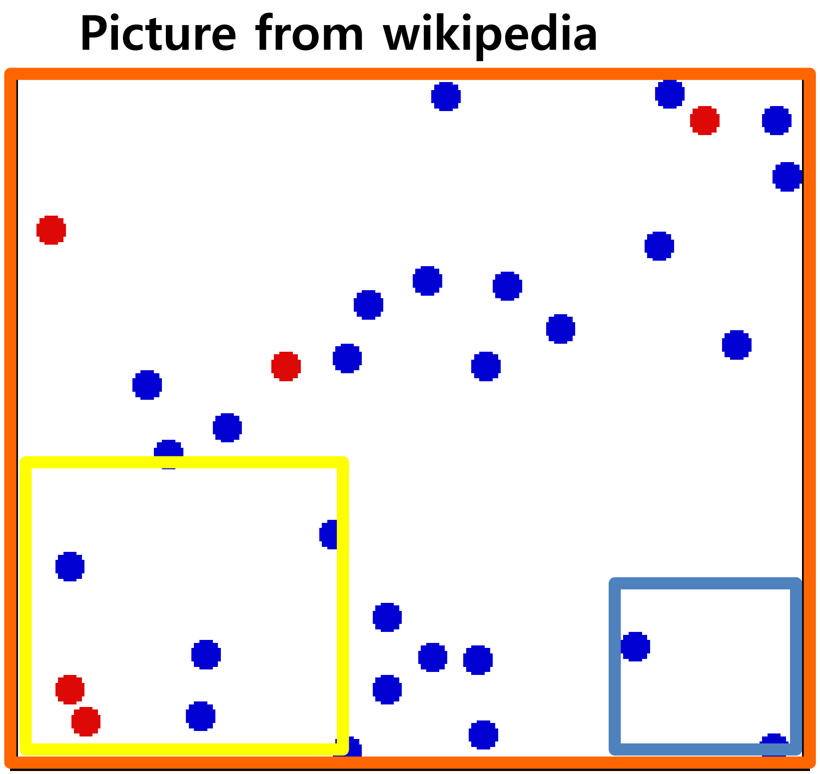
- Communications between methamaticians and physicisits will be beneficial/crucial to understand the M-gap? T? …Uncountable Infinities?

Gravity on Matter Equation of State

Hyeong-Chan Kim
(KNUT)

APCTP Mini-symposium
Pohang, Korea,
July. 22, 2016.

- **Newtonian gravity:**
  In preparation, H.K., Gungwon Kang (KISTI),
- **General relativity:**
  In Preparation, H.K., Chueng Ji (NCSU).



Statistical description is

possible.

possible??

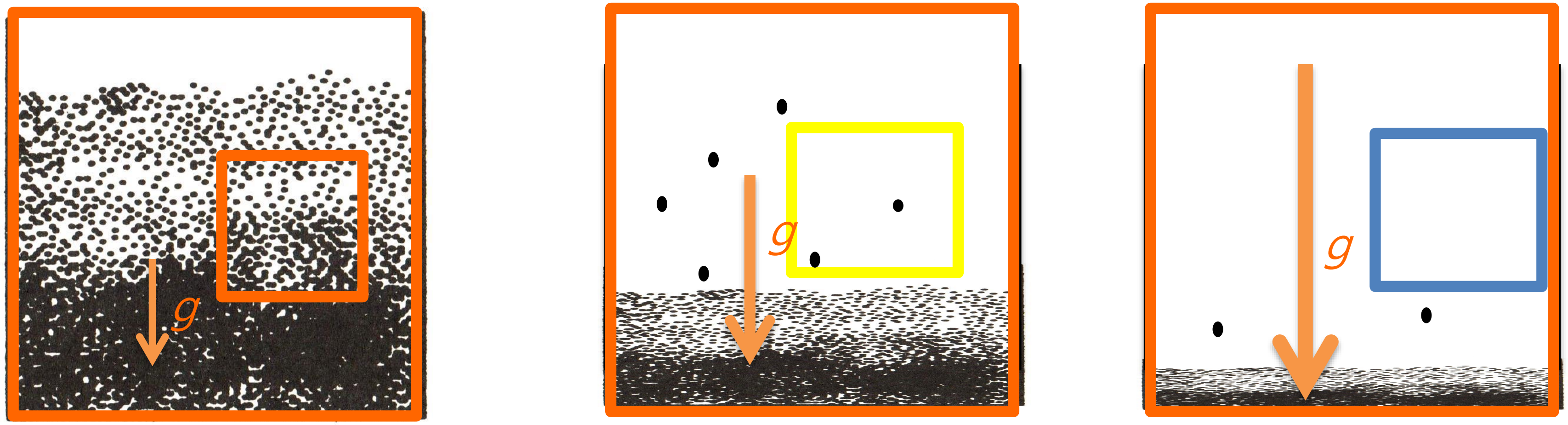
impossible.

Statistical description requires enough number of particles in a box.

This restricts the size of the box for a given system of density.

$$nV \sim N_A$$

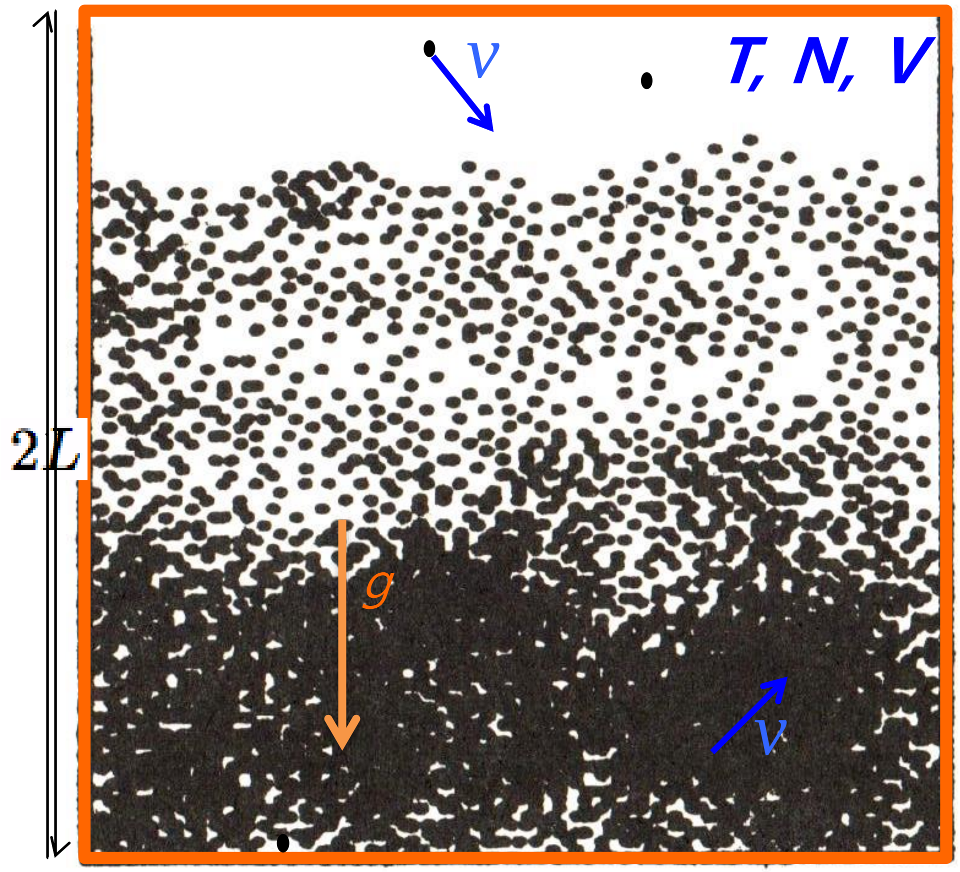
Statistical description in a gravity



Impossible. Local description in terms of pressure and density is inappropriate.

A strong gravity restricts the region of space where the statistical description is possible.

$$H = \frac{1}{2}\mu_0 v^2 + \mu_0 g z, \qquad -L \le z \le L,$$



$$n(z) \propto e^{-\beta\mu_0 g z}$$

$$P(z) = n(z) k_B T,$$

The density and pressure are position dependent.

Kinetic energy:

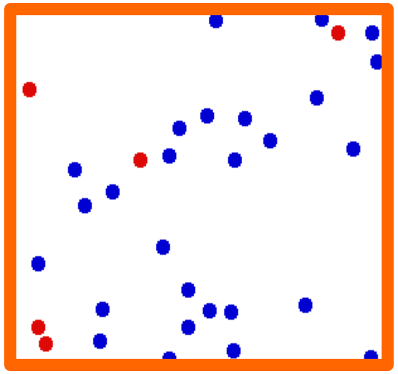
$$K = 3Nk_BT/2,$$

Therefore, the avarage speed of a particle is independent of its height.

For the time being, we assume the volume contains statistically enough number of particles.

Q: It appears genuine that strong gravity affects on the distributions of matters.
Then, how is the Equation of State?

Two EoS we are interested in:



EOS 1: $PV = Nk_BT,$ **(Ideal gas)**

Adiabatic ideal gas:

$$dS = 0,$$

$$dT = -PdV/C_V, \leftarrow dU = TdS - PdV,$$

$$PdV + VdP = -\frac{Nk_B}{C_V}PdV \Longrightarrow \boxed{P = K\rho^\gamma;} \quad \gamma = \frac{C_V + Nk_B}{C_V},$$

EOS 2: **Polytropic EoS**

$$dP(r) = -\rho(r)g(r)dr,$$

Balance equation + EoS2 provides star structure in Newtonian gravity.

Statistics (Summary)

Basic principle:

The number of particles in unit phase volume is proportional to

$$n(x^i, p_j) \propto e^{-\beta H(x^i, p_j)}.$$

Partition function:

$$\begin{cases} \log Z_N = N \log Z_1 - \log N! \,, \\ \log Z_1 \equiv \log \left[\left(\frac{\mu_0}{h} \right)^3 \int_V d^3x \int d^3v \, e^{-\beta H} \right] \end{cases}$$

Total energy and entropy:

$$U_N(T, X) \equiv - \left[\frac{\partial \log Z_N}{\partial \beta} \right]_V$$

$$\frac{S_N}{N k_B} \equiv \frac{U_N}{N k_B T} + N^{-1} \log Z_N(X)$$

Heat Capacity:

$$C_V \equiv \frac{\partial U_N}{\partial T}$$

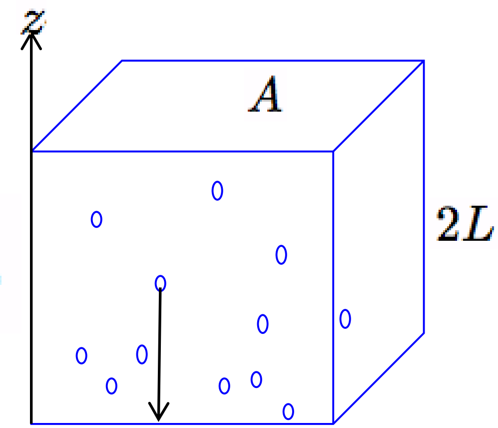
distribution of ptls:

$$n(z, v) = \frac{N}{Z_1} \left(\frac{\mu_0}{h} \right)^3 e^{-\beta H}.$$

N-particle system in a box:

One particle Hamitonian:

$$H = \frac{1}{2}\mu_0 v^2 + \mu_0 g z, \qquad -L \leq z \leq L,$$



One particle partition function:

Landsberg, et. al. (1994).

$$\log Z_1 \equiv \log\left[\left(\frac{\mu_0}{h}\right)^3 \int_V d^3x \int d^3v\, e^{-\beta H}\right] = \log\frac{V}{(\hbar/(\mu_0 c))^3} + \frac{3}{2}\log\frac{k_B T}{2\pi\mu_0 c^2} + \log\frac{\sinh X}{X}.$$

$$V = 2L \times A,$$

Order parameter for gravity:

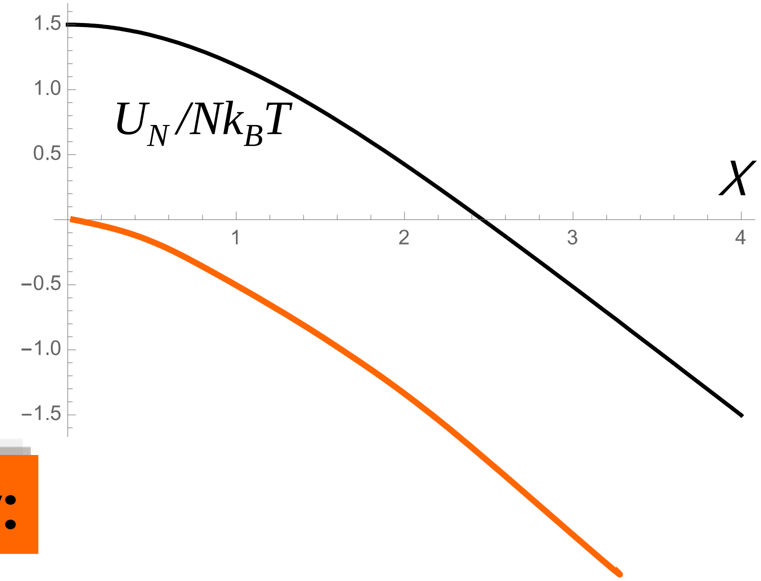
$$X \equiv \frac{M\mathcal{G}}{N k_B T} = \frac{MgL}{N k_B T} \geq 0; \qquad \mathcal{G} \equiv gL, \quad M = N\mu_0.$$

Ratio btw the grav. Potential energy to the thermal kinetic ener gy.

Ideal Gas in Constant Gravity

Internal energy:

$$U_N(T,X) \equiv -\left[\frac{\partial \log Z_N}{\partial \beta}\right]_V = \left(\frac{5}{2} - X\coth(X)\right) N k_B T.$$



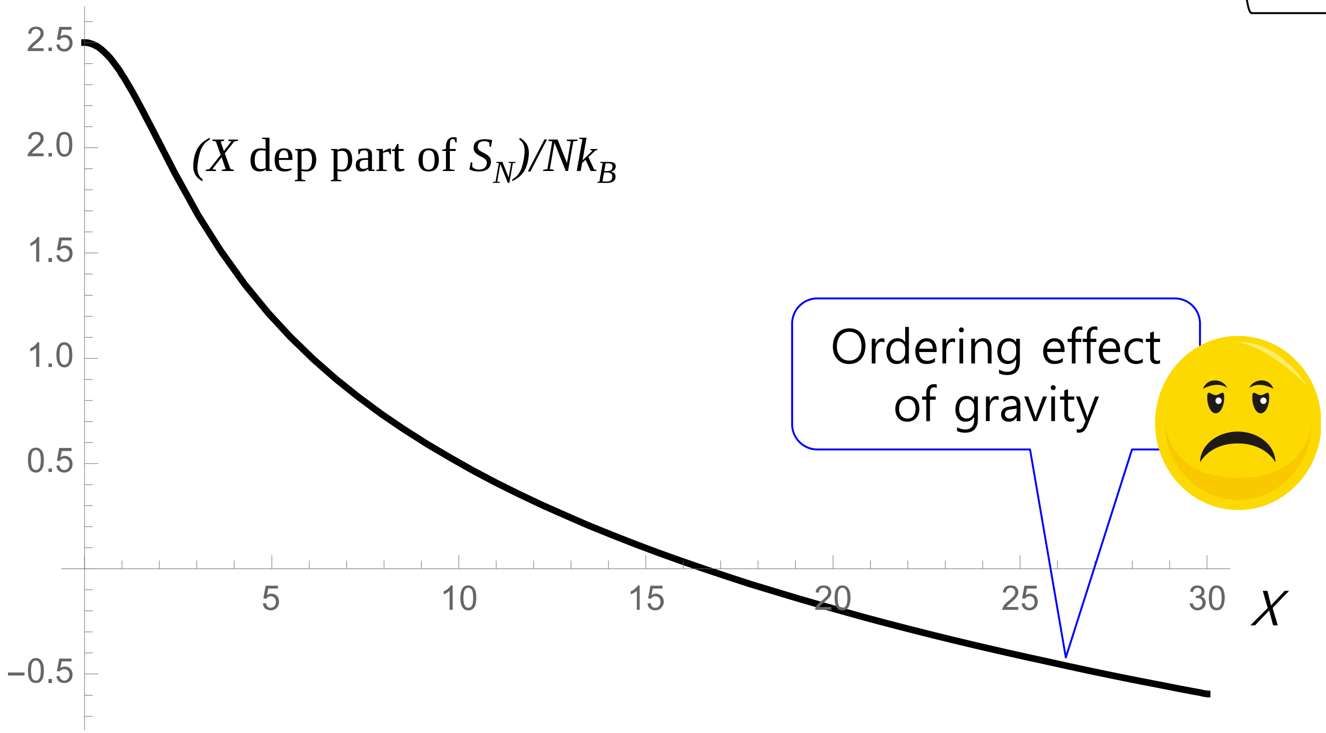
Gravitational potential energy:

$$\Omega = U_N - K = N k_B T(1 - X\coth X),$$

Ideal Gas in Constant Gravity

Entropy:

$$\frac{S_N}{Nk_B} \equiv \frac{U_N}{Nk_BT} + N^{-1}\log Z_N(X) = \frac{7}{2} + \log\frac{V/N}{(\hbar/\mu_0 c)^3} + \frac{3}{2}\log\frac{k_BT}{2\pi\mu_0 c^2} + \underbrace{\log\left(\frac{\sinh X}{X}\right) - X\coth X}_{\times}.$$



Entropy takes negative values for large gravity.

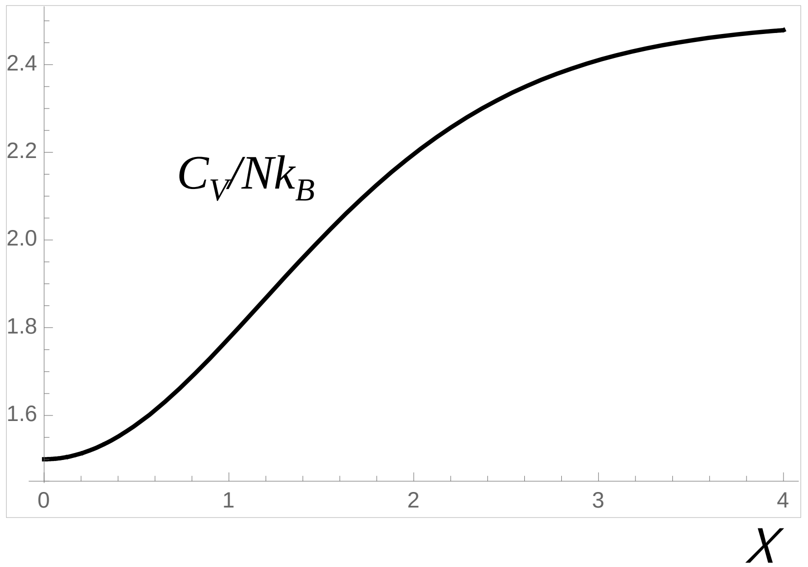
Heat Capacities:

Heat capacity for constant gravity:

$$C_V \equiv \frac{\partial U_N}{\partial T} = C_{V,0} + Nk_B\left(1 - \frac{X^2}{\sinh^2 X}\right)$$

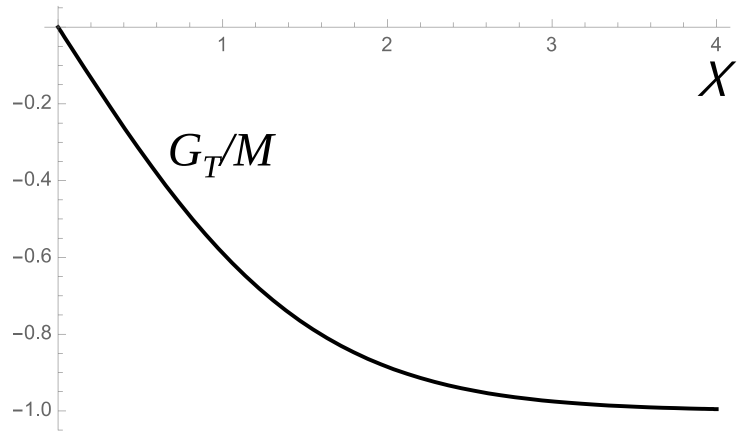
Monatomic gas

$$C_{V,0} = 3Nk_B/2.$$



Gravity capacity for constant *T*:

$$\frac{G_T}{M} \equiv \frac{1}{M}\frac{\partial \Omega}{\partial \mathcal{G}} = \frac{X}{\sinh^2 X} - \coth X.$$



EoS 1 in Constant Gravity

Distribution of particles is position dependent:

$$n(z) \equiv \int d^3v\, n(z,v) = \frac{N}{V}\frac{X}{\sinh X} e^{-\beta\mu_0 g z}.$$

$$P(z) \equiv \int_{v_z>0} d^3v(2p_z)v_z n(z,v) = \bar{P}\frac{X}{\sinh X} e^{-\frac{\mu_0 g z}{k_B T}} = n(z)k_B T,$$

However, local and the averaged values satisfy the ideal gas law.

$$\bar{P} \equiv \frac{1}{2L}\int_{-L}^{L} P(z)dz = \frac{N k_B T}{V}$$

Pressure difference: (in the zero size limit, it becomes the balance equation)

$$\Delta P \equiv P(L) - P(-L) = -2X\bar{P} = -\frac{Mg}{A},$$

The new EoS 2 in the adiabatic case

First law: $dU_N = k_B T dS - \bar{P} dV + \Omega (d \log \mathcal{G}).$

The energy is dependent on both of the temperature and gravity:

$$dU_N = C_V dT + G_T d\mathcal{G} = \frac{C_V}{N k_B}(V d\bar{P} + \bar{P} dV) + G_T d\mathcal{G},$$

Adiabaticity: $dS = 0,$

We get,

$$\frac{3}{2}\frac{d\bar{P}}{\bar{P}} + \frac{5}{2}\frac{dV}{V} = \left(1 - \frac{X^2}{\sinh^2 X}\right)\frac{dX}{X}.$$

$$\bar{P} = \bar{K}(X)\rho^{5/3}; \qquad \bar{K} \equiv K\left(\frac{X e^{X \coth(X) - 1}}{\sinh X}\right)^{2/3}. \qquad \rho \equiv M/V$$

EoS appears to be factorized. However,

$$X = \frac{M\mathcal{G}}{N k_B T} = \frac{MgL}{\bar{P}V} = \frac{\rho g L}{\bar{P}} = \frac{Mg/A}{2\bar{P}}$$

contains thermodynamic variables.

The new EoS 2: Limiting behaviors

Weak gravity limit: $\bar{P} \approx K\rho^{5/3}\left[1 + \frac{1}{9}\left(\frac{\mathcal{G}}{K\rho^{2/3}}\right)^2 + \cdots\right].$

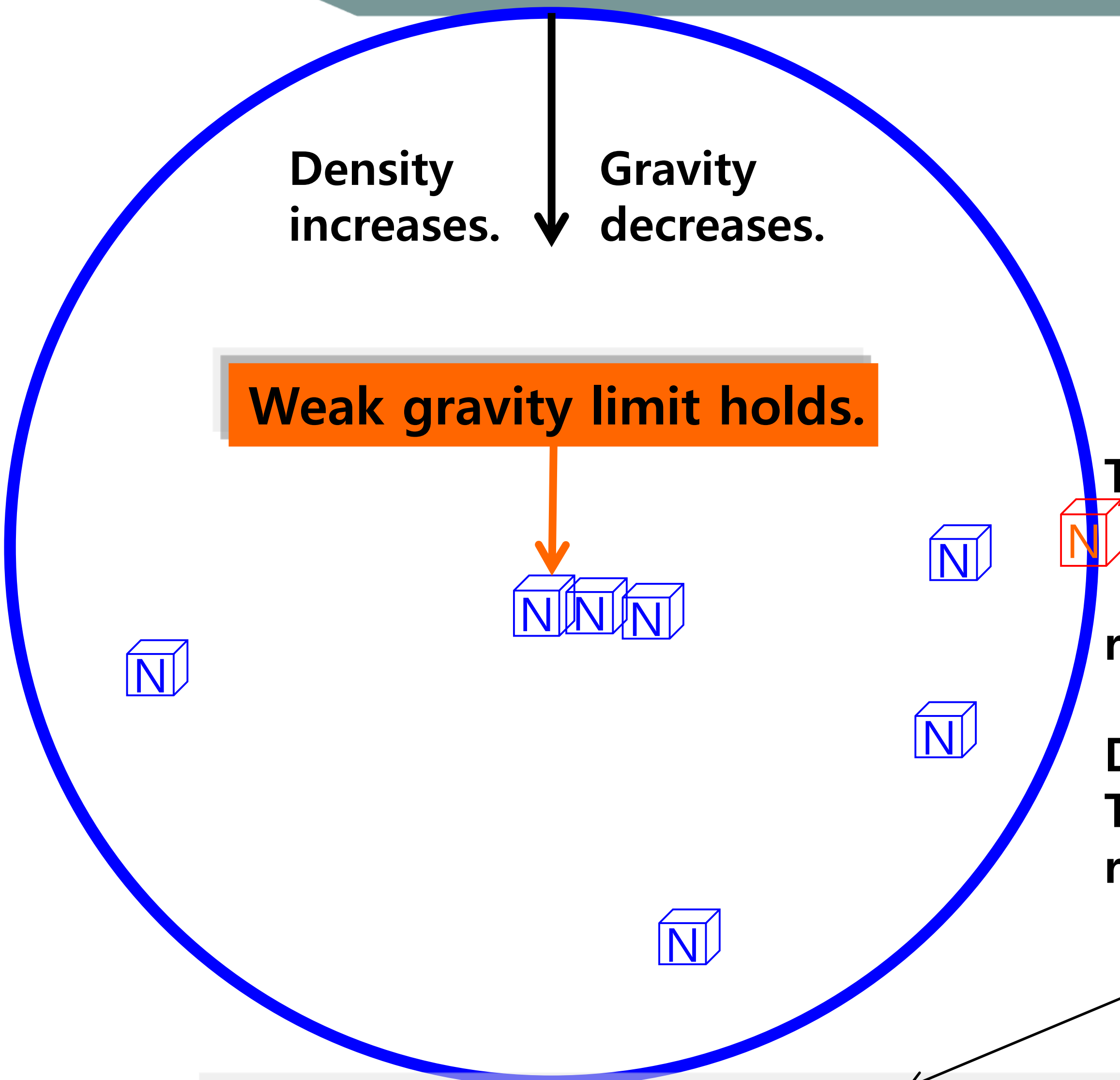
The correction is second order.
<u>The pressure difference equation</u> becomes the balance equation.
Therefore, one can ignore this correction in the small size limit of the system.

Therefore, the gravity effects on EOS is negligible if the system size is small.

Strong gravity (macroscopic system) limit: $\bar{P} \approx K^{3/5}\left(\frac{2M\mathcal{G}}{e}\right)^{2/5}\rho^{7/5}.$

shows noticeable difference even in the non-relativistic, Newtonian regime: $k_B T < \mu_0 g L \ll \mu_0 c^2.$
Pressure difference equation becomes a discrete difference eq.

An application of the modified EOS



Strong gravity limit hold s.

The pressure difference eq

$$\Delta P/\bar{P} = -2X.$$

replaces the balance equation.

Density increases exponentially. Therefore, this strong gravity regime is thin.

Therefore, for most stars in astrophysics, the gravity effects on EOS is ignorable.

Then, when can we observe the gravity effect?

A: Only when the macroscopic effects are unavoidable.

Macroscopic: size > kinetic energy/gravitational force

Every cases are beyond
the scope of
the classical Newtonian theory.

Relativistic Case

Equation of State in General Relativity

General Covariance:

Freely falling frame = locally flat
EOS in freely falling frame = EOS in flat ST

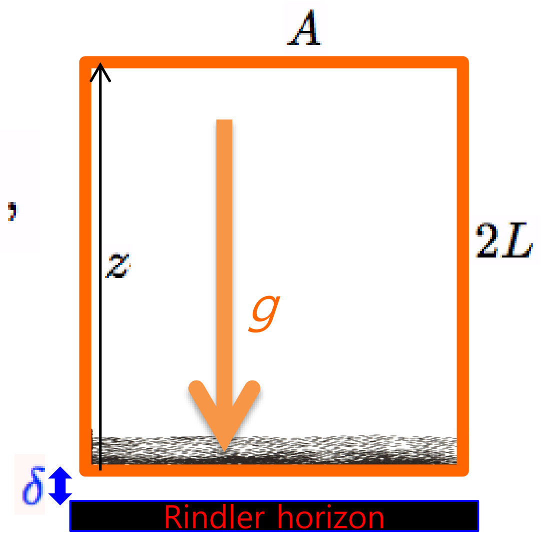
Scalar quantity

Density, pressure, temperature are scalar quantities. Therefore, their values in other frame must be the same as those in the freely falling frame.

Generalization: Ideal Gas in Constant Gravity

N-particle system in a box (Rindler spacetime):

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(1+gz)^2 dt^2 + dx^2 + dy^2 + dz^2,$$



Hamiltonian:

$$H \equiv \sum p_i v^i - L = \mu_0 \left(1+gz\right)\sqrt{1+\frac{p^2}{\mu_0^2}}.$$

Partition function:

Temperature at z=0

$$\log Z_N[V,\beta,\mathcal{G}] = N \log \frac{V}{(\hbar/\mu_0)^3 N} + N + \frac{3N}{2}\log\frac{k_B T}{2\pi\mu_0} + N\log\left[\frac{(\beta\mu_0)^{3/2}}{\sqrt{\pi/2}}\,\mathcal{Z}\right].$$

$$\mathcal{Z}(\alpha_+,\alpha_-) = \frac{1}{2X}\left[-\frac{K_1(\alpha_+)}{\alpha_+} + \frac{K_1(\alpha_-)}{\alpha_-}\right],$$

$\alpha_\pm = \beta\mu_0(1\pm\mathcal{G})$ and $X \equiv \frac{\alpha_+ - \alpha_-}{2}$.

Continuity Equation:

The Continuity Equation:

$$\frac{\partial P}{\partial x^0} = 0, \qquad \frac{\partial P}{\partial x^j} = -(\rho + P)\frac{\partial \log\sqrt{-g_{00}}}{\partial x^j} \quad \Rightarrow \quad \left(\frac{1}{g} + z\right)\frac{\partial P}{\partial z} = -(\rho + P).$$

$$\longrightarrow \quad P(z) = -\frac{g}{1+gz}\int^z dz'\rho(z').$$

The number and energy densities:

$$n(z,p) = \frac{N}{h^3 Z_1}e^{-\beta H}$$

$$n(z) \;=\; \frac{N}{V\mathcal{Z}}\frac{K_2(\alpha)}{\alpha}; \qquad \alpha = \beta\mu_0\,(1+gz)\,.$$

$$\rho(z) \equiv \frac{1}{1+gz}\int d^3p\,H(z,p)n(z,p) \;=\; -\frac{M_0}{V\mathcal{Z}}\left(\frac{\partial}{\partial\alpha}\frac{K_2(\alpha)}{\alpha}\right).$$

$$P(z) = \frac{M_0}{V\mathcal{Z}}\frac{K_2(\alpha)}{\alpha^2} = \frac{n(z)k_BT}{1+gz}. \longrightarrow \boxed{P(z) = n(z)k_BT(z)}$$

Ideal gas law is satisfied locally (not globally) by local temperature
.

Total energy and Entropy:

Define pressure in Rindler space:

$$p(z) \equiv (1+gz)P(z) = n(z)k_B T$$

$$p_{\rm avg} = \frac{1}{V}\int d^3r\, p(z) = \frac{Nk_BT}{V}, \qquad \Delta p = -2(u+1)X p_{\rm avg}.$$

Ideal gas law is satisfied on the whole system if one define an average pressure for Rindler space.

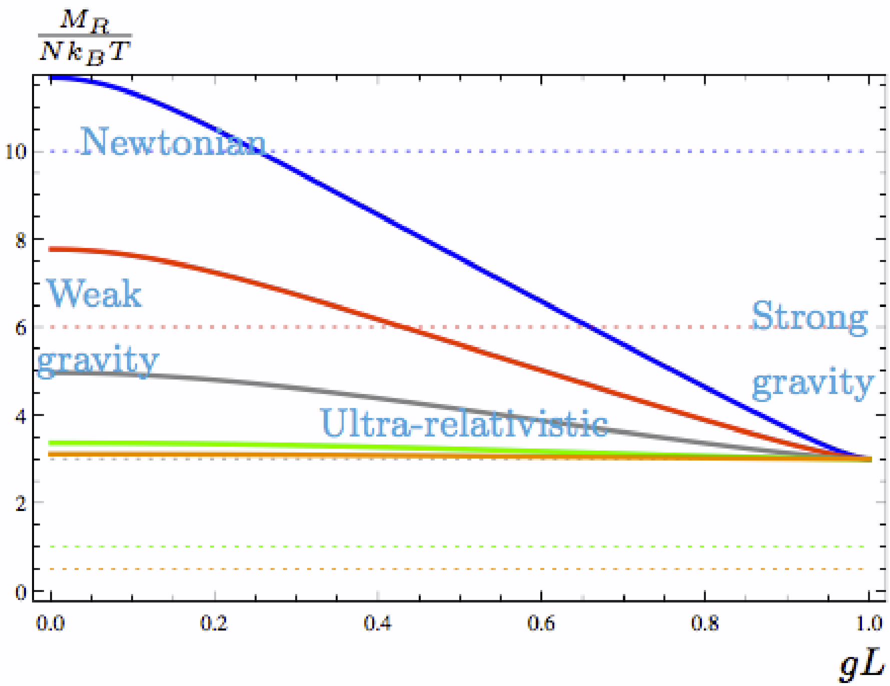
Pressure difference relation is modified.

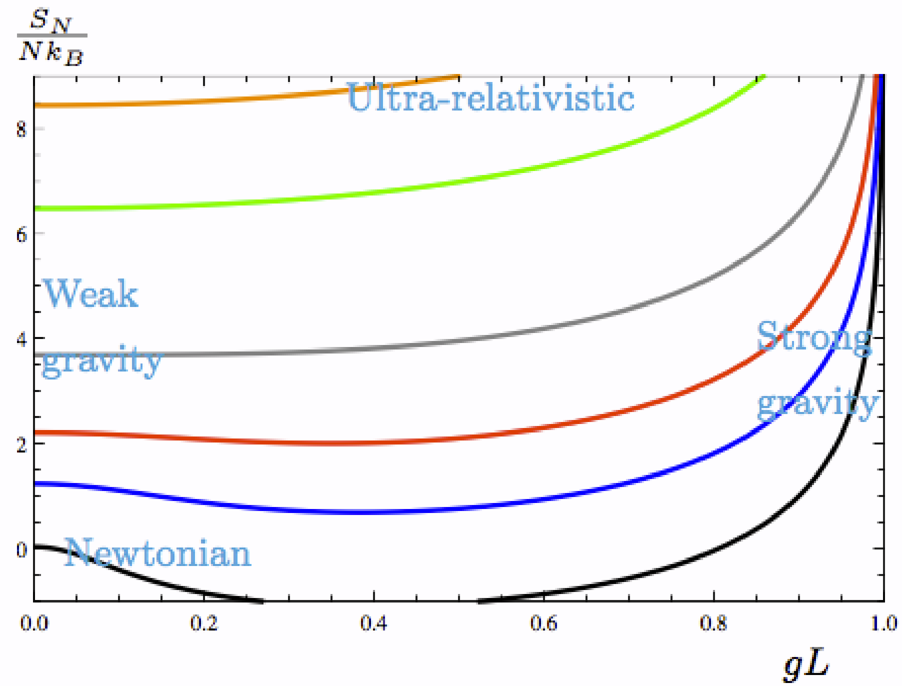
The total energy and entropy in Rindler frame:

$$M_R(T,\mathcal{G}) \equiv -\left(\frac{\partial \log Z_N}{\partial\beta}\right)_V = Nk_BT\, m(\alpha_+,\alpha_-),$$

$$\frac{S_N(V,T,\mathcal{G})}{Nk_B} \equiv \frac{M_R}{Nk_BT} + N^{-1}\log Z_N = \log\frac{eV/N}{2\pi^2(\hbar/\mu_0)^3} + s(\alpha_+,\alpha_-).$$

Total energy and Entropy





$$m(\alpha_+, \alpha_-) \equiv 1 - \frac{K_2(\alpha_+) - K_2(\alpha_-)}{2X\mathcal{Z}},$$

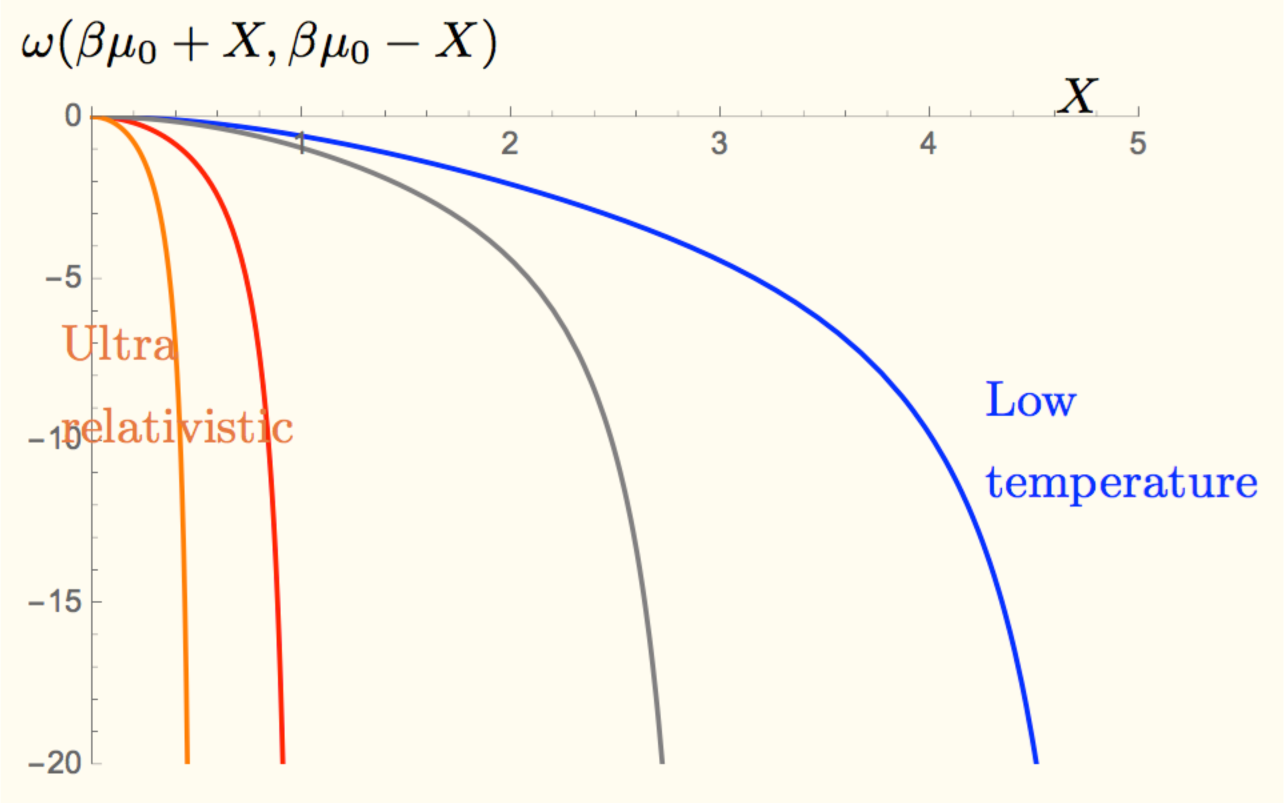
$$s(\alpha_+, \alpha_-) \equiv m(\alpha_+, \alpha_-) + \log \mathcal{Z}.$$

Gravitational potential:

Gravitational potential Energy:

$$\Omega \equiv M_R - U_{\rm int} - M_0 = N k_B T\, \omega(\alpha_+, \alpha_-),$$

$$\omega(\alpha_+, \alpha_-) \equiv 1 - \frac{1}{2\mathcal{Z}} \left(\frac{K_2(\alpha_+)}{\alpha_+} + \frac{K_2(\alpha_-)}{\alpha_-} \right)$$

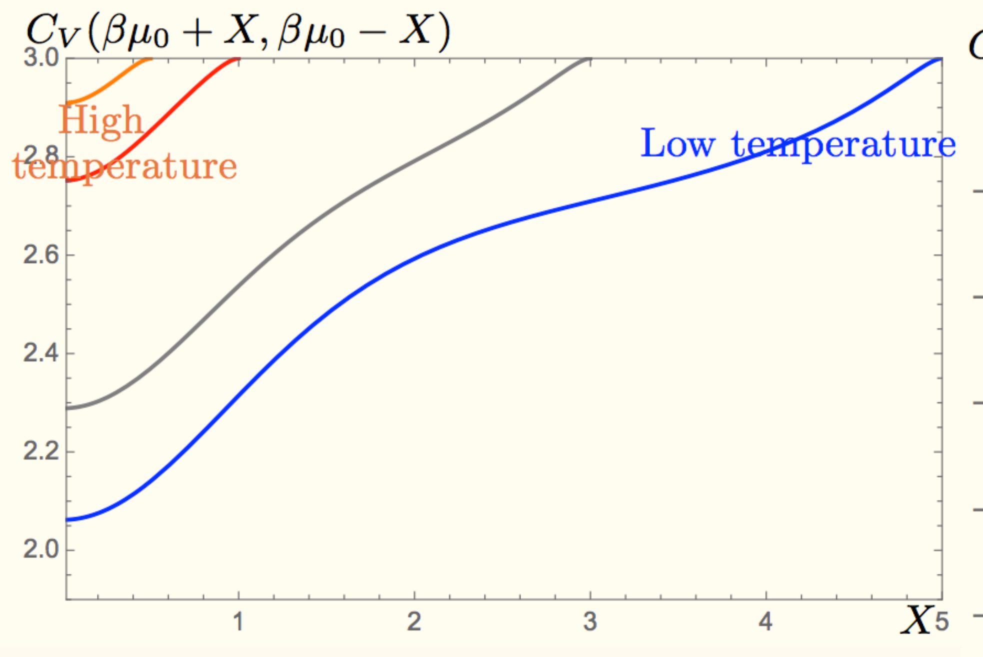


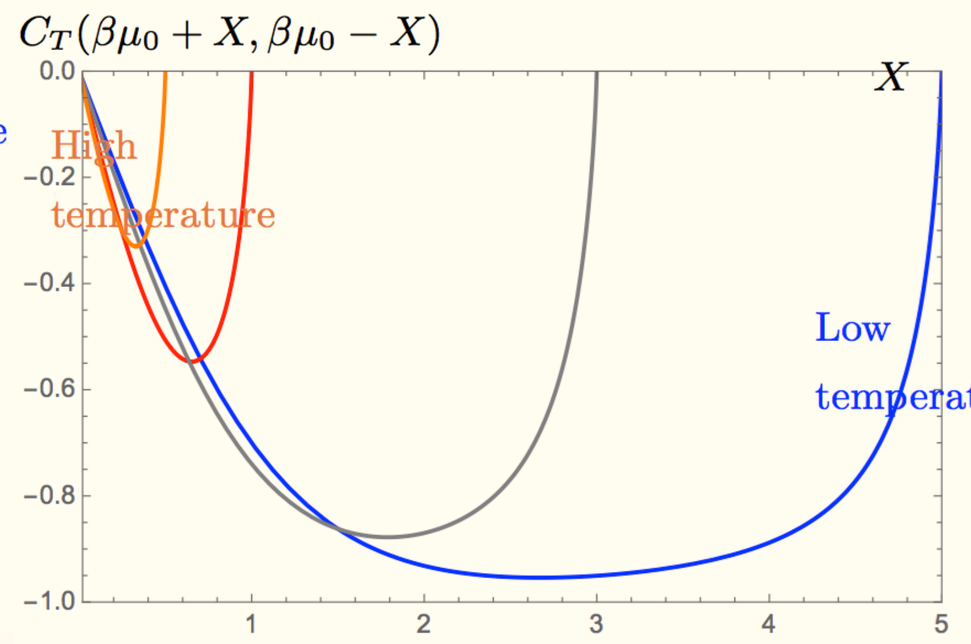
Heat Capacities:

Heat Capacities for constant volume, gravity and for constant volume, temperature:

$$\frac{C_V}{Nk_B} \equiv \frac{1}{Nk_B}\left(\frac{\partial M_R}{\partial T}\right)_{V,\mathcal{G}} = m(\alpha_+,\alpha_-) - \mu_0\beta\partial_S m(\alpha_+,\alpha_-) - X\partial_A m(\alpha_+,\alpha_-),$$

$$\frac{C_T}{M_0} \equiv M_0^{-1}\left(\frac{\partial M_R}{\partial \mathcal{G}}\right)_{T,V} = \partial_A m,$$





Strong gravity regime:

$gL/c^2 = 1$ corresponds to the case that the bottom of the box touches the event horizon

Parameterize the distance from the event horizon as

$$1 - \mathcal{G} = 1 - gL = g(g^{-1} - L) = g\delta,$$

Radiation?

$$M_R \approx N k_B T \Big[3 + \Big(- \log \frac{\mu_0 g \delta}{2 k_B T} - \gamma - K_0(2\beta\mu_0) \Big) (\beta\mu_0)^2 \times (g\delta)^2 \Big].$$

$$S_N \approx A\, n_S \left[\log \left(\frac{e^4}{4\pi^2 (\hbar/\mu_0)^3 g n_S} \right) - 3 \log \frac{\mu_0}{k_B T} - 2 \log(g\delta) + \cdots \right].$$

Area proportionality

Thermodynamic first law

Differentiating the definition of entropy $S_N/k_B \equiv M_R/k_BT + \log Z_N$

$$dM_R = k_BTdS_N + M_R\frac{dT}{T} - k_BTd\log Z_N .$$

From the functional form of the partition function:

$$d\log Z_N = \frac{M_R}{k_BT}\frac{dT}{T} + \frac{N}{V}dV - N\omega(\alpha_+,\alpha_-)\,d\log\mathcal{G}$$

Combining the two, we get the first law:

$$dM_R = k_BTdS_N - \frac{Nk_BT}{V}dV + \Omega\,d\log\mathcal{G}.$$

Gravitational potential energy

Equation of state for an adiabatic system

From the first law with dS=0, $dM_R = -Nk_BTd\log V + \Omega d\log \mathcal{G}.$

From the definition of Heat capacities. $dM_R = C_V dT + C_T d\mathcal{G} = Nk_BT\left[\frac{C_V}{Nk_B} d\log(Nk_BT) + (X\partial_A m)\, d\log \mathcal{G}\right].$

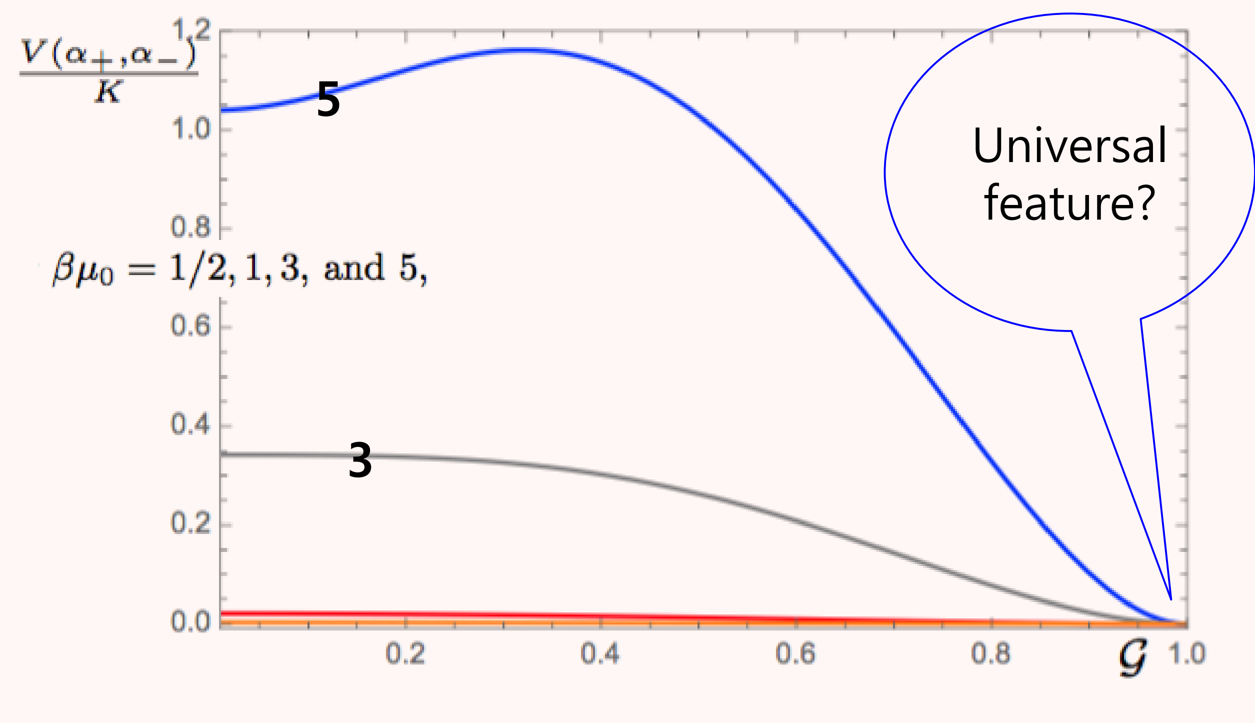
Combining the two, we get:

$$\frac{dV}{V} = \mathcal{C}_{\mathcal{G}}\, d\log X - \mathcal{C}\, d\log(Nk_BT),$$

$$\mathcal{C}_{\mathcal{G}} \equiv \omega - X\partial_A m. \qquad \mathcal{C} \equiv \frac{C_V}{Nk_B} - \omega + X\partial_A m = \beta\mu_0(u + 1 - \partial_S m).$$

Integrating:

$$V = K\,\frac{e^{-m(\alpha_+,\alpha_-)}}{\mathcal{Z}(\alpha_+,\alpha_-)}.$$



Newtonian gravity limit:

$$V = \left(\sqrt{\frac{8}{\pi}}\frac{K}{e^{3/2}}\right)\left(\frac{M_0}{Nk_BT}\right)^{3/2}\frac{X}{\sinh X}e^{X\coth X-1} + O(\beta\mu_0)^{1/2}.$$

Reproduce the Newtonian result.

Strong gravity limit:

$$V \approx \frac{2K}{e^3}\delta^2 g^2(\beta\mu_0)^3 \longrightarrow k_BT = \frac{2\pi}{e}\left(\frac{\tilde{K}}{\tilde{A}}\right)^{1/3}\times\left(\frac{\hbar g}{2\pi c}\right).$$

$V = 2AL \approx 2Ag^{-1}$ $\tilde{K} \equiv K/(\hbar/\mu_0 c)^3$ $\tilde{A} \equiv A/\delta^2$

Unruh temperature?

At present, we cannot determine the dimensionless part.

Conclusion

Newtonian gravity:

	Locally	Macroscopically
$PV = Nk_BT,$	Kept	kept
$P = K\rho^{\gamma}$	kept	modified

Rindler spacetime:

EoS in **the strong gravity limit** appears to determine the temperature of the system to be that of the **Unruh temperature**.

Future plan

1) System around a blackhole horizon

2) Quantum mechanical effect?

3) Self gravitating system?

4) Relation to blackhole thermodynamics?

5) Dynamical system?

5) Etc…

Thanks, All Participants.